

FIXED-ENERGY MULTI-PARTICLE MSA IMPLIES DYNAMICAL LOCALIZATION

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ABSTRACT. This work is a continuation of [15] where we described two elementary derivations of the variable-energy MSA bounds from their fixed-energy counterparts, in the framework of single-particle disordered quantum particle systems on graphs with polynomially bounded growth of balls. Here the approach of [15] is extended to multi-particle Anderson Hamiltonians with interaction; it plays a role similar to that of the Simon–Wolf criterion for single-particle Hamiltonians. A simplified, fixed-energy multi-particle MSA scheme was developed in our earlier work [7], based on a multi-particle adaptation of techniques from Spencer’s paper [36]. Combined with a simplified variant of the Germinet–Klein argument [29] described in [14], the outcome of the fixed-energy analysis results in an elementary proof of multi-particle dynamical localization with the decay of eigenfunction correlators faster than any power-law.

1. INTRODUCTION.

This manuscript is an extended version of my talk given recently at the workshop ”*Mathematics of quantum disordered systems*” organized by the *Institut de Mathématiques de Jussieu* at Chevaleret (Paris) and by the *Institut Galilée* at Villetaneuse. A reader familiar with [15] may notice that substantial portions of the text are borrowed from that preprint, often by the ”copy-n-paste” method. A good time-saving tool, it is also a well-known source of a.s. creation of notational inconsistencies, for which I sincerely apologize in advance. In my talk, this text has been promised to be uploaded promptly to **arXiv**; a more detailed version will be made available later.

I would like to stress here a point which may have been insufficiently emphasized in the talk: although several approaches to the derivation of the spectral and dynamical localization from the FEMSA (fixed-energy multi-scale analysis) estimates for **single-particle** systems are known by now¹ (cf., e.g., [5], [30]), this text focuses only on two methods which seem to provide the shortest path to the multi-particle localization from the FEMPMSA results, which are much simpler to obtain than their variable-energy analogs, the VEMPMSA (variable-energy multi-particle MSA) bounds. The choice is certainly biased by the author’s personal preferences, and there is no doubt that alternative, more efficient and more general methods will appear in near future.

¹I thank Peter Hislop for a fruitful exchange on this subject during the above mentioned workshop ”*Mathematics of quantum disordered systems*”.

Several ideas employed in the derivation " $\text{FEMPMSA} \Rightarrow \text{VEMPMSA}$ " have been used by other researchers in the context of single-particle systems. The "disorder-energy" measurable space $\Omega \times \mathbb{R}$ appears already in the work by Martinelli–Scoppola [35]. In a more general context, it becomes the main scene of action in the Simon–Wolf paper [37] (see [37] for a more extensive bibliography and a discussion of a series of works which initiated the research by Simon and Wolf). The fact that the matrix elements of resolvents are rational functions with a "moderate" number of poles is explicitly used, e.g., by Bourgain–Kenig [5], in a "hard" situation (Bernoulli–Anderson Hamiltonians). Germinet–Klein [30] use spectral reductions for random Hamiltonians with virtually no assumption on the marginal probability distribution of the random potential (the scatterers amplitude of an alloy potential may follow any probability law not concentrated on a single point). While the singular nature of the random potential forces one to make use of technically involved analytic and probabilistic tools, making a strong assumption on the regularity of the marginal distributions (viz., sufficient regularity of the probability *density*) gives rise to a significant simplification of spectral reductions ($\text{FEMSA} \Rightarrow \text{VEMSA}$).

This list can be continued ...

Needless to say that the Fractional Moment Analysis (FMM) always starts as a fixed-energy analysis.

In the framework of the multi-particle Anderson Hamiltonians, the above mentioned strong assumption on the regularity of the marginal *density* remains so far the only means to achieve more optimal, physically reasonable bounds on long-range charge transfer processes (tunneling) in an interacting quantum system of $N \geq 3$ particles; cf. a brief discussion in subsection 2.3. Resonances occurring in such systems can be qualified as "structural"; they do not appear in 1-particle systems and can be treated in a relatively simple way in 2-particle systems. The solution to this problem (explicitly analyzed by Aizenman and Warzel [3]), proposed in our works [11], [12], requires the above mentioned regularity assumption. Therefore, limiting the spectral reduction ($\text{FEMSA} \Rightarrow \text{VEMSA}$) to random Hamiltonians obeying this assumption does not seem to be an overly big concession – at least, until a new, more efficient solution is found to the problem of long-range tunneling due to structural resonances in systems with $N > 2$ particles.

On the other hand, proving merely the Anderson localization phenomenon in an interacting quantum system with an arbitrary (but fixed) finite number of particles is a simpler task than proving *efficient* decay bounds for eigenfunction correlators. This can be done under a much weaker hypothesis of Hölder (or even log-Hölder) continuity of the *cumulative* marginal probability distribution function (PDF) of the external random potential featuring the IID, IAD (=Independence At Distance) or an (appropriate) strong mixing property. The main geometrical tool here is the notion of "separability" of pairs of finite volumes developed in our joint works with Yuri Suhov [8, 9] (for $N = 2$ particles) and in [10] (for $N \geq 2$ particles).

2. BASIC NOTATIONS, FACTS AND ASSUMPTIONS

Throughout this paper, we work with discrete Schrödinger operators (DSO) acting in Hilbert spaces of square-summable complex functions on connected countable graphs. Indeed, the techniques and results of the MSA, initially developed for operators on periodic lattices, are naturally extended to more general graphs with polynomially bounded growth of balls (such graphs as Bethe lattices remain so far

out of the MSA's reach). Another motivation for presenting the new approach on a graph comes from the fact that the natural language for the description of a system of $N > 1$ interacting indistinguishable quantum particles (bosons or fermions) is that of a symmetric power of the configuration space \mathcal{Z} of the respective single-particle system; already in the case where the configuration space is $\mathcal{Z} = \mathbb{Z}^d$, $d > 1$, its N -th *symmetric* power is no longer a periodic lattice.

2.1. Graphs, configurations and graph Laplacians. Consider a finite or countable connected graph $(\mathcal{G}, \mathcal{E})$, with the set of vertices \mathcal{G} and the set of edges \mathcal{E} ; for brevity, we will often call \mathcal{G} the graph, omitting the reference to \mathcal{E} . We denote by $d_{\mathcal{G}}(\cdot, \cdot)$ (sometimes simply by $d(\cdot, \cdot)$) the canonical distance on the graph \mathcal{G} : $d_{\mathcal{G}}(x, y)$ is the length of the shortest path $x \rightsquigarrow y$ over the edges. We will assume that the growth of balls $B_L(x) := \{y : d_{\mathcal{G}}(x, y) \leq L\}$ is polynomially bounded:

$$\sup_{x \in \mathcal{G}} |B_L(x)| \leq C_d L^d, \quad L \geq 1. \quad (2.1)$$

In particular, the coordination number $n_{\mathcal{G}}(x) := \{y : d_{\mathcal{G}}(x, y) = 1\}$ of any vertex x is bounded by C_d (even by $C_d - 1$).

Given a connected graph $(\mathcal{Z}, \mathcal{E}_{\mathcal{Z}})$ serving as the configuration space of quantum particles, the configuration space of a system of $N > 1$ distinguishable particles is the cartesian product \mathcal{Z}^N ; it is usually endowed with the following graph structure: a pair (\mathbf{x}, \mathbf{y}) is an edge iff, for some $j_o \in [1, N]$, $(x_{j_o}, y_{j_o}) \in \mathcal{E}_{\mathcal{Z}}$, while for all $i \neq j_o$ one has $x_i = y_i$. In other words, \mathbf{y} is obtained from \mathbf{x} by moving exactly one particle to one of its nearest neighbors in \mathcal{Z} . To make explicit this choice, we use the boldface notations \mathcal{Z}^N for the vertex set of the N -particle configuration graph and $\mathcal{E}^{(N)}$ for the described edge set. In general, boldface notations will be reserved for "multi-particle" objects.

Introduce the following mapping from \mathcal{Z}^n , $n \geq 1$, to the collection of finite subsets of \mathcal{Z} :

$$\Pi : \mathbf{x} = (x_1, \dots, x_n) \mapsto \{x_1, \dots, x_n\}. \quad (2.2)$$

We will call $\Pi \mathbf{x}$ the *support* of the configuration $\mathbf{x} \in \mathcal{Z}^n$. (Note that in the framework of indistinguishable particles, only $\Pi \mathbf{x}$ would be physically observable.) Similarly, the support of a "polydisk" $\mathbf{B}_L^{(n)}(\mathbf{x}) = \times_{j=1}^n B_L(x_j)$ is the set

$$\Pi \mathbf{B}_L(\mathbf{x}) = \cup_{j=1}^n B_L(x_j) \subset \mathcal{Z}. \quad (2.3)$$

Further, given a non-empty index subset $\mathcal{J} \subseteq [1, N] \cap \mathbb{Z}$, define a partial projection (or partial support)

$$\Pi_{\mathcal{J}} : (x_1, \dots, x_N) \mapsto \{x_j, j \in \mathcal{J}\} \subset \mathcal{Z}.$$

For $\mathcal{J} = \emptyset$ set, formally, $\Pi_{\emptyset} \mathbf{x} = \emptyset$.

Apart from the graph distance $\mathbf{d}(\cdot, \cdot) = d_{\mathcal{Z}^N}(\cdot, \cdot)$ on \mathcal{Z}^N , it will be convenient to use the max-distance $\boldsymbol{\rho}$ and symmetrized max-distance $\boldsymbol{\rho}_S$ defined as follows:

$$\boldsymbol{\rho}(\mathbf{x}, \mathbf{y}) = \max_{1 \leq j \leq N} d_{\mathcal{Z}}(x_j, y_j), \quad \boldsymbol{\rho}_S(\mathbf{x}, \mathbf{y}) = \min_{\pi \in \mathfrak{S}_N} \boldsymbol{\rho}(\mathbf{x}, \pi(\mathbf{y})),$$

where the elements $\pi \in \mathfrak{S}_N$ of the symmetric group \mathfrak{S}_N act on vertices $\mathbf{x} \in \mathcal{Z}^N$ by permutations of the coordinates. In terms of $\boldsymbol{\rho}$, a polydisk $\mathbf{B}_L^{(n)}(\mathbf{x})$ is a ball of radius L centered at \mathbf{x} . (If $\mathcal{Z} = \mathbb{Z}^d$ with max-distance, then polydisks are cubes.)

The canonical (negative) graph Laplacian $(-\Delta_{\mathcal{G}})$ on a finite or countable graph $(\mathcal{G}, \mathcal{E})$ is given by

$$(-\Delta_{\mathcal{G}}f)(x) = \sum_{\langle x, y \rangle} (f(x) - f(y)) = n_{\mathcal{G}}(x)f(x) - \sum_{\langle x, y \rangle} f(y) \quad (2.4)$$

where we use a popular notation $\langle x, y \rangle$ for a pair of nearest neighbors $x, y \in \mathcal{G}$, i.e., $d_{\mathcal{G}}(x, y) = 1$, and $n_{\mathcal{G}}(x)$ is the coordination number of the point x . For brevity, we will sometimes use slightly abusive notations like $\langle x, y \rangle \in \Lambda$, $\Lambda \subset \mathcal{G}$ instead of $\langle x, y \rangle \in (\Lambda \times \Lambda) \cap \mathcal{E}_{\mathcal{G}}$.

From this point on, unless otherwise specified, we will use the notation \mathcal{G} only for finite connected graphs, either in the single-particle context or in a situation where the nature of a graph (single- or multi-particle) is irrelevant, while \mathcal{Z} will stand for a countable connected graph with polynomial growth of balls. Finite connected subgraphs of \mathcal{Z}^N will be sometimes denoted by \mathcal{G} .

In operator form, we can write, for an arbitrary (connected) graph \mathcal{G} ,

$$-\Delta_{\mathcal{G}} = n_{\mathcal{G}} - \sum_{\langle x, y \rangle} \Gamma_{x, y}, \quad \Gamma_{x, y} = |\mathbf{1}_x\rangle\langle\mathbf{1}_y|,$$

where $n_{\mathcal{G}}$ is the operator of multiplication by the function $x \mapsto n_{\mathcal{G}}(x)$. Given a subgraph $\Lambda \subsetneq \mathcal{G}$, define its internal, external and the so-called edge boundary (relative to \mathcal{G}) as follows:

$$\begin{aligned} \partial_{\mathcal{G}}^- \Lambda &= \{y \in \Lambda : d_{\mathcal{G}}(y, \mathcal{G} \setminus \Lambda) = 1\}, \quad \partial_{\mathcal{G}}^+ \Lambda = \partial_{\mathcal{G}}^- \mathcal{G} \setminus \Lambda, \\ \partial_{\mathcal{G}} \Lambda &= \{(x, y) \in \partial_{\mathcal{G}}^- \Lambda \times \partial_{\mathcal{G}}^+ \Lambda : d_{\mathcal{G}}(x, y) = 1\}. \end{aligned}$$

Working with a given graph $\mathcal{G}(\subset \mathcal{Z})$, we always mean by a ball $B_R(u) \subset \mathcal{G}$ the set $\{y \in \mathcal{G} : d_{\mathcal{G}}(u, y) \leq R\}$, i.e., the **ball relative to the metric space** $(\mathcal{G}, d_{\mathcal{G}})$.

The Laplacian (hence, a DSO) in a subgraph $\Lambda \subset \mathcal{G}$ can be defined in various ways. The two most popular choices are:

- The canonical (negative) Laplacian in Λ , $(-\Delta_{\Lambda}^N f) = (-\Delta_{\Lambda} f)$, defined as in (2.4) with \mathcal{G} replaced by Λ . In this context, it is usually considered as an analog of the Neumann Laplacian, and reads as follows:

$$(-\Delta_{\Lambda}^N f)(x) = n_{\Lambda}(x) - \sum_{\langle x, y \rangle \in \Lambda} f(y). \quad (2.5)$$

- The Dirichlet Laplacian $(-\Delta_{\Lambda, \mathcal{G}}^D) = \mathbf{1}_{\Lambda}(-\Delta_{\Lambda, \mathcal{G}}^D) \mathbf{1}_{\Lambda} \upharpoonright \ell^2(\Lambda)$. Here we use a natural injection $\ell^2(\Lambda) \hookrightarrow \ell^2(\mathcal{G})$. The Dirichlet counterpart of (2.6) is

$$(-\Delta_{\Lambda}^D f)(x) = n_{\mathcal{G}}(x) - \sum_{\langle x, y \rangle \in \Lambda} f(y), \quad (2.6)$$

with $n_{\mathcal{G}}(x) \geq n_{\Lambda}(x)$, so $(-\Delta_{\Lambda}^D) \geq (-\Delta_{\Lambda}^N)$ in the sense of quadratic forms.

We will use the Dirichlet Laplacians and DSO H_{Λ}^D . Given a decomposition $\mathcal{G} = \Lambda \sqcup \Lambda^c$, $\Lambda^c := \mathcal{G} \setminus \Lambda$, we can write

$$\begin{aligned} -\Delta_{\mathcal{G}}^D &= n_{\mathcal{G}} - \sum_{\langle x, y \rangle \in \Lambda} \Gamma_{x, y} - \sum_{\langle x, y \rangle \in \Lambda^c} \Gamma_{x, y} - \sum_{\langle x, y \rangle \in \partial \Lambda} (\Gamma_{x, y} + \Gamma_{y, x}) \\ &= ((-\Delta_{\Lambda}^D) \oplus (-\Delta_{\Lambda^c}^D)) - \Gamma_{\Lambda, \mathcal{G}} \end{aligned}$$

with $\Gamma_{\Lambda, \mathcal{G}} = \sum_{\langle x, y \rangle \in \partial \Lambda} (\Gamma_{x, y} + \Gamma_{y, x})$. Respectively for the DSO $H_{\mathcal{G}} = -\Delta_{\mathcal{G}}^{\text{D}} + V$, where $V : \mathcal{G} \rightarrow \mathbb{R}$ is usually referred to as the potential, one has

$$H_{\mathcal{G}} = H_{\mathcal{G}, \Lambda}^{\bullet} - \Gamma_{\Lambda, \mathcal{G}}, \quad H_{\mathcal{G}, \Lambda}^{\bullet} := (-\Delta_{\Lambda}^{\text{D}} + V) \oplus (-\Delta_{\Lambda^c}^{\text{D}} + V).$$

We omit the superscript "N", since the nature of the boundary conditions in \mathcal{G} is not related to the choice of Dirichlet or Neumann decoupling induced by $\mathcal{G} = \Lambda \sqcup \Lambda^c$.

The spectrum of a (finite-dimensional) operator $H_{\mathcal{G}}$, i.e., the set of its eigenvalues (EVs) counting multiplicities, will be denoted by $\Sigma(H_{\mathcal{G}})$. The resolvent of a Hamiltonian $\mathbf{H}_{\Lambda}(\omega)$, $\Lambda \subseteq \mathcal{Z}^N$, will be denoted by $\mathbf{G}_{\Lambda}(E; \omega)$ and its matrix elements in the delta-basis (Green functions) by $\mathbf{G}_{\Lambda}(\mathbf{x}, \mathbf{y}; E; \omega)$. The subscript Λ will be omitted when $\Lambda = \mathcal{Z}^N$.

In a number of formulae and statements, we will use parameters $\beta, \tau, \varrho \in (0, 1)$, and $\alpha \in (1, 2)$. Unless otherwise specified, we assume that $\beta = 1/2$, $\tau = 1/8$, $\varrho = (\alpha - 1)/2 = 1/6$ and $\alpha = 3/2$. Note that the exponent $\frac{1+\varrho}{\alpha}$ figuring in Definition 2.1 then equals $7/8$.

The function $\gamma : (m, L) \mapsto m(1 + L^{-\tau})$ introduced below is a convenient replacement for the decay exponent ("mass") used in the MSA and dependent upon the scale. We merely make this dependence explicit. Clearly, $\gamma(m, L) > m$ for any $L > 0$.

Definition 2.1. Given numbers $E \in \mathbb{R}$, $m > 0$ and $L \in \mathbb{N}^*$, a ball $\mathbf{B}_L(\mathbf{u})$ is called

- E -resonant (E -R, in short), if $\text{dist}(\Sigma(\mathbf{H}_{\mathbf{B}_L(\mathbf{u})}), E) < e^{-L^{\beta}}$, and E -nonresonant (E -NR), otherwise;
- (E, m) -nonsingular ((E, m) -NS), if for all $\mathbf{x}, \mathbf{y} \in \mathbf{B}_L(\mathbf{u})$ with $\rho(\mathbf{x}, \mathbf{y}) \geq L^{\frac{1+\varrho}{\alpha}}$

$$|\partial \mathbf{B}_L(\mathbf{u})| \cdot |\mathbf{G}_{\mathbf{B}_L(\mathbf{u})}(\mathbf{x}, \mathbf{y}; E)| \leq e^{-\gamma(m, L)\rho(\mathbf{x}, \mathbf{y})}, \quad (2.7)$$

where

$$\gamma(m, L) := m(1 + L^{-\tau}), \quad (2.8)$$

and (E, m) -nonsingular ((E, m) -NS), otherwise.

2.2. Geometric resolvent inequality. The second resolvent identity implies the so-called Geometric resolvent equation for the resolvents $G_{\mathcal{G}}(E) = (H_{\mathcal{G}} - E)^{-1}$, $G_{\Lambda^c}(E) = (H_{\Lambda^c} - E)^{-1}$, $G_{\mathcal{G}, \Lambda}^{\bullet}(E) = (H_{\mathcal{G}, \Lambda}^{\bullet} - E)^{-1}$:

$$G_{\mathcal{G}}(E) = G_{\mathcal{G}}^{\bullet}(E) + G_{\mathcal{G}}^{\bullet}(E) \Gamma_{\Lambda, \mathcal{G}} G_{\mathcal{G}}(E). \quad (2.9)$$

For $x, u \in \Lambda$ and $y \in \Lambda^c$, one has $G_{\mathcal{G}}^{\bullet}(x, u; E) = G_{\Lambda}^{\text{D}}(x, u; E)$ and $G_{\mathcal{G}}^{\bullet}(x, y; E) = 0$. This results in the Geometric resolvent equation for the Green functions

$$G_{\mathcal{G}}(x, y; E) = \sum_{\langle u, u' \rangle \in \partial_{\mathcal{G}} \Lambda} G_{\Lambda}^{\text{D}}(x, u; E) G_{\mathcal{G}}(u', y; E) \quad (2.10)$$

and the Geometric resolvent inequality (GRI)

$$|G_{\mathcal{G}}(x, y; E)| \leq \sum_{\langle u, u' \rangle \in \partial_{\mathcal{G}} \Lambda} |G_{\Lambda}^{\text{D}}(x, u; E)| |G_{\mathcal{G}}(u', y; E)|. \quad (2.11)$$

In the rest of the paper, the superscript "D" will be omitted, since we do not use the Neumann boundary conditions.

2.3. Assumptions on the random potential. The efficiency of the quantitative bounds on eigenfunctions and eigenfunction correlators for $N \geq 3$ interacting particles available at the moment and the complexity of the proofs depend upon the assumptions on the random potential.

For clarity of presentation, we always assume that the random potential field $V : \mathcal{Z} \times \Omega \rightarrow \mathbb{R}$ on a graph \mathcal{Z} is IID.

The first, more general condition (leading to weaker results) is as follows:

(W1): *The marginal probability distribution function (PDF) F_V is uniformly Hölder continuous: there are constants $C_H \in (0, +\infty)$, $\delta \in (0, 1]$ such that*

$$\sup_{t \in \mathbb{R}} (F_V(t+s) - F_V(t)) \leq C_H s^\delta. \quad (2.12)$$

To obtain more optimal decay bounds on eigenfunction correlators, with the help of a method developed in [12]–[11], one needs an additional assumption which we will describe now. Introduce the following notations. Given a finite subset $\Lambda \subset \mathcal{Z}$, let $\xi_\Lambda(\omega)$ be the sample mean of the random field V over the Λ ,

$$\xi_\Lambda(\omega) = |\Lambda|^{-1} \sum_{x \in \Lambda} V(x, \omega)$$

and define the "fluctuations" of V relative to the sample mean, $\eta_x(\omega) = V(x, \omega) - \xi_\Lambda(\omega)$, $x \in \Lambda$. Denote by $\mathfrak{F}_{\eta, \Lambda}$ the sigma-algebra generated by $\{\eta_x : x \in \Lambda\}$, and by $F_{\xi_\Lambda}(\cdot | \mathfrak{F}_{\eta, \Lambda})$ the conditional distribution function of ξ_Λ given $\mathfrak{F}_{\eta, \Lambda}$. Assuming that $\Lambda \subset \mathcal{Z}$ and $\text{diam}(\Lambda) \leq R$, introduce, for $s \geq 0$, an $\mathfrak{F}_{\eta, \Lambda}$ -measurable random variable

$$\nu_{\xi_\Lambda}(s; \omega) := \text{ess sup} \sup_{t \in \mathbb{R}} |F_{\xi_\Lambda}(t+s | \mathfrak{F}_{\eta, \Lambda}) - F_{\xi_\Lambda}(t | \mathfrak{F}_{\eta, \Lambda})|. \quad (2.13)$$

The stronger assumption on the random field V is as follows:

(W2): *There exist $C', C'', A', A'', B', B'' \in (0, +\infty)$ such that for any finite subset $\Lambda \subset \mathcal{Z}$ with $\text{diam}(\Lambda) \leq R$,*

$$\forall s \in (0, 1] \quad \mathbb{P} \{ \nu_{\xi_\Lambda}(s; \omega) \geq C R^A s^B \} \leq C' R^{A'} s^{B'}. \quad (2.14)$$

In the case where the random field V is IID, a more natural condition should refer to the cardinality $|\Lambda|$ of the subset Λ , but we will use the above property only in a situation where Λ is a subset of some ball, of an explicitly bounded diameter.

For further use, define the following function on $[0, 1]$:

$$h_L = h_L^{(N, \mathbf{x}, \mathbf{y})} : s \mapsto |\mathbf{B}_L^{(N)}(\mathbf{x})| \cdot |\mathbf{B}_L^{(N)}(\mathbf{y})| C L^A s^B + C' L^{A'} s^{B'}. \quad (2.15)$$

See the discussion of the validity of (W2) in subsection 8.2 below.

2.4. Assumptions on the interaction potential. For brevity, we consider only finite-range interactions \mathbf{U} generated by a pair interaction potential $U^{(2)}(\cdot)$:

(U1): *There is an integer $r_0 \geq 0$ and a function $U^{(2)} : \mathbb{N} \rightarrow \mathbb{R}$ with finite support, $\text{supp } U^{(2)} \subset [0, r_0]$, such that*

$$\mathbf{U}(\mathbf{x}) = \sum_{1 \leq i < j \leq N} U^{(2)}(d(x_i, x_j)). \quad (2.16)$$

The parameter r_0 will be called the range of the interaction \mathbf{U} .

In the case where the single-particle configuration space is a group (e.g., \mathbb{Z}^d or \mathbb{R}^d), the above form of \mathbf{U} corresponds to translation invariant interactions; in the case of a Euclidean space, such an interaction is also isotropic. In all cases, \mathbf{U} of the form (2.16) is permutation symmetric. We stress that neither of these properties is crucial for the multi-particle MSA scheme. Moreover, this scheme can be easily adapted to interactions with a hard core.

The assumption of finite range simplifies the induction on particles. In [13], we described an extension of the variable-energy MPMSA to interactions decaying at an exponential or subexponential rate,

$$|U^{(2)}(r)| \leq c_1 e^{-c_2 r^{c_3}}$$

with $c_1, c_2 \in (0, +\infty)$ and $c_3 > 0$ sufficiently close to 1. The proof of the fixed-energy MPMSA bounds can follow essentially the same path. In fact, the most significant modification is required in the proof of Lemma 6.4; see the details in [13]. In the framework of the fixed-energy MPMSA (which is simpler), we plan to address infinite-range interactions in a forthcoming paper.

3. MAIN RESULTS

Let \mathcal{B}_1 be the set of all bounded Borel functions f with $\|f\|_\infty \leq 1$.

Theorem 3.1. *Fix any integer $\hat{N} \geq 2$. Under the assumptions (W1) and (U1), for any $N \in [1, \hat{N}]$ there exists $g_0(N) < +\infty$ such that for all $|g| \geq g_0(N)$ and some $m = m(g, N) \geq \text{Const}(N) \ln |g| > 0$:*

(A) *with probability one, the random operator $\mathbf{H}(\omega) = \mathbf{H}_0 + g\mathbf{V}(\omega) + \mathbf{U}$ has pure point spectrum and all eigenfunctions $\Psi_j(\omega)$ rapidly decaying at infinity: for each Ψ_j and some $\hat{\mathbf{x}}_j$, for all \mathbf{x} and some $C(\hat{\mathbf{x}}_j, \omega) \in (0, +\infty)$,*

$$|\Psi_j(\mathbf{x}, \omega)| \leq C(\hat{\mathbf{x}}_j, \omega) e^{-a \ln^{1+c} \rho(\mathbf{x}, \hat{\mathbf{x}}_j)}; \quad (3.1)$$

(B) *for all points \mathbf{x}, \mathbf{y} and some $a, c, C(\mathbf{x}) \in (0, +\infty)$,*

$$\mathbb{E} \left[\sup_{f \in \mathcal{B}_1} |\langle \mathbf{1}_{\mathbf{y}} | f(\mathbf{H}(\omega)) | \mathbf{1}_{\mathbf{x}} \rangle| \right] \leq C(\mathbf{x}) e^{-a \ln^{1+c} \rho(\mathbf{x}, \mathbf{y})}. \quad (3.2)$$

With $f = f_\lambda : t \mapsto e^{-it\lambda}$, $\lambda \in \mathbb{R}$, Eqn (3.2) reads as the strong dynamical localization for the random Hamiltonians $\mathbf{H}^{(N)}(\omega)$.

Technically, we prove upper bounds in (3.1)-(3.2) of the form $L_k^{-\kappa(1+\theta)^k}$ at distances $\sim L_k \sim (L_0)^{\alpha^k}$, which can be easily translated into the above decay rate.

Assertion (A) can be made stronger, without replacing the hypothesis (W1) by a stronger condition (W2). Indeed, one can prove an exponential decay of all eigenfunctions, using the variable-energy MPMSA developed in [10]. In the present paper, we focus on a simpler approach, based upon the fixed-energy MPMSA.

Note also that the decay bounds 3.1–3.2 have an unusual form as compared to their more traditional counterparts from the single-particle MSA: constants $C(\hat{\mathbf{x}}_j, \omega)$, $C(\mathbf{x})$ depend upon the multi-particle configuration \mathbf{x} (actually, upon the diameter of its support, $\text{diam}(\{x_1, \dots, x_N\})$). This feature is present in the first papers on multi-particle localization [10], [3]; we discuss it below in more detail.

As was already said, in order to obtain more optimal localization bounds, viz.: exponential spectral localization (as in [10]) and uniform estimates on the decay of eigenfunctions and their correlators, we make use of a stronger assumption (W2).

Theorem 3.2. *Fix any integer $\hat{N} \geq 2$. Under the assumptions (W2) and (U1), for any $N \in [1, \hat{N}]$ there exists $g_0(N) < +\infty$ such that for all $|g| \geq g_0(N)$ and some $m = m(g, N) \geq \text{Const}(N) \ln |g| > 0$:*

(A) *with probability one, the random operator $\mathbf{H}(\omega) = \mathbf{H}_0 + g\mathbf{V}(\omega) + \mathbf{U}$ has pure point spectrum and all eigenfunctions $\Psi_j(\omega)$ decay exponentially fast at infinity: for each Ψ_j , some $\hat{\mathbf{x}}_j(\omega) \in \mathcal{Z}^N$, $C_j(\omega) \in (0, +\infty)$ and all $\mathbf{x} \in \mathcal{Z}^N$,*

$$|\Psi_j(\mathbf{x}, \omega)| \leq C_j(\omega) e^{-m\rho_S(\mathbf{x}, \hat{\mathbf{x}}_j)}; \quad (3.3)$$

(B) *for all points \mathbf{x}, \mathbf{y} and some $a, c, C \in (0, +\infty)$,*

$$\mathbb{E} \left[\sup_{f \in \mathcal{B}_1} |\langle \mathbf{1}_{\mathbf{y}} | f(\mathbf{H}(\omega)) | \mathbf{1}_{\mathbf{x}} \rangle| \right] \leq C e^{-a \ln^{1+c} \rho_S(\mathbf{x}, \mathbf{y})}. \quad (3.4)$$

It is readily seen that, given a point $\mathbf{u} \in \mathcal{Z}^N$ (e.g., $\mathbf{u} = \hat{\mathbf{x}}_j$),

$$\rho(\mathbf{x}, \mathbf{u}) \geq \rho(\mathbf{x}, \{\pi(\mathbf{u}), \pi \in \mathfrak{S}_N\}) \xrightarrow{\rho(\mathbf{x}, \mathbf{u}) \rightarrow +\infty} +\infty,$$

so that the assertions (A) and (B) of Theorem 3.2 imply their counterparts from Theorem 3.1 (under a more restrictive hypothesis (W2)). An important point is that Theorem 3.2 provides a more precise and explicit information on the decay properties of the EFs and their correlators. Moreover, once formulated in a physically relevant language of *indistinguishable* particles (bosons or fermions), Theorem 3.2 operates with a *natural distance* ρ_S between particle configurations. Possible lack of decay in terms of the non-symmetrized distance ρ is due to the fact that any \pm -symmetric eigenfunction $\Psi(\mathbf{x})$ takes the values of the same magnitude along an orbit $\mathcal{O}(\mathbf{x}) := \{\pi(\mathbf{x}), \pi \in \mathfrak{S}_N\}$.

Working with restrictions of the Hamiltonian $\mathbf{H}^{(N)}(\omega)$ to the subspaces of symmetric or antisymmetric functions on \mathcal{Z}^N would be simpler on the N -th symmetric power of the 1-particle (physical) configuration space \mathcal{Z} . In the framework of the variable-energy scaling analysis, such an approach has been used in our earlier work [13]. Here, for the sake of brevity, we prefer to work with distinguishable particles, i.e., with Hamiltonians $\mathbf{H}^{(N)}(\omega)$ in the entire Hilbert space $\ell^2(\mathcal{Z}^N)$. This also leads, formally, to more general results. Nevertheless, it is to be stressed again that only the language of indistinguishable particles is physically relevant.

Finally, note that adaptations of the variable-energy MPMSA to localization near the lower edge of the spectrum (without the assumption of strong disorder) and to weakly interacting systems ("sufficiently" localized without interaction) have been obtained by Ekanga [21, 22], in the case where $\mathcal{Z} = \mathbb{Z}^d$, under the assumption of log-Hölder continuity of the PDF F_V of the external random potential.

Theorems 3.1 and 3.2 give the final results of the multi-particle localization analysis. However, the novelty of the present work resides essentially in Theorems 9.2 and 9.3, deriving the spectral and dynamical localization from the results of the fixed-energy analysis of resolvents.

4. SUBHARMONICITY ON GRAPHS

The results of this section apply to arbitrary graphs, including \mathcal{Z}^N , $N \geq 2$.

Definition 4.1. Let \mathcal{G} be a finite connected graph, $L \geq \ell \geq 0$ two integers and $q \in (0, 1)$. A function $f : \mathcal{G} \rightarrow \mathbb{R}_+$ is called (ℓ, q) -subharmonic in a ball $B_L(u) \subsetneq \mathcal{G}$ if for any ball $B_\ell(x) \subseteq B_L(u)$ one has

$$f(x) \leq q \max_{y \in B_{\ell+1}(x)} f(y). \quad (4.1)$$

We will often use the notation $\mathcal{M}(f, \Lambda) := \max_{x \in \Lambda} |f(x)|$.

Lemma 4.1. *If a function $f : \mathcal{G} \rightarrow \mathbb{R}_+$ defined on a finite connected graph \mathcal{G} is (ℓ, q) -subharmonic in a ball $B_L(x) \subsetneq \mathcal{G}$, with $L \geq \ell \geq 0$, then*

$$f(x) \leq q^{\lfloor \frac{L+1}{\ell+1} \rfloor} \mathcal{M}(f, \mathcal{G}) \leq q^{\frac{L-\ell}{\ell+1}} \mathcal{M}(f, \mathcal{G}). \quad (4.2)$$

In fact, the factor $\mathcal{M}(f, \mathcal{G})$ in the RHS of (4.2) can be replaced by $\mathcal{M}(f, B_{L+1}(x))$.

Proof. See [14]. \square

Lemma 4.2. *Consider a finite connected graph \mathcal{G} and a ball $B_L(u) \subsetneq \mathcal{G}$, with $L \geq \ell \geq 0$. Fix numbers $E \in \mathbb{R}$, $m > 0$ and suppose that all balls $B_\ell(x)$ inside $B_L(u)$ are (E, m) -NS. Then $\forall y \in \mathcal{G} \setminus B_L(u)$ the function*

$$f : x \mapsto |G_{\mathcal{G}}(x, y; E)|$$

is (ℓ, q) -subharmonic in $B_L(u)$ with $q = e^{-\gamma(m, \ell)\ell}$.

Proof. The claim follows directly from the Definition 4.1. \square

Lemma 4.1 suffices to assess the Green functions in a ball $B_L(u)$ which does not contain any singular ℓ -ball, but to analyze the situation where $B_L(u)$ does not contain any *pair* of disjoint singular ℓ -balls, one needs the following extension of Lemma 4.1, inspired by the proof of Theorem 1 in [36]: approaching a single "bad" ball separately from the points x and y .

Lemma 4.3. *Let \mathcal{G} be a finite connected graph, and $f : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{R}_+$, $f : (x, y) \mapsto f(x, y)$, be a function which is separately (ℓ, q) -subharmonic in $x \in B_{r'}(u') \subset \mathcal{G}$ and in $y \in B_{r''}(u'') \subset \mathcal{G}$, with $r', r'' \geq \ell \geq 0$ and $d(u', u'') \geq r' + r'' + 2$. Then*

$$f(u', u'') \leq q^{\lfloor \frac{r'+1}{\ell+1} \rfloor + \lfloor \frac{r''+1}{\ell+1} \rfloor} \mathcal{M}(f, \mathcal{G} \times \mathcal{G}) \leq q^{\frac{r'+r''-2\ell}{\ell+1}} \mathcal{M}(f, \mathcal{G} \times \mathcal{G}). \quad (4.3)$$

Proof. See [14]. \square

5. EIGENVALUE CONCENTRATION BOUNDS

5.1. One-volume EVC bounds. The first eigenvalue concentration (EVC) bound required for our scheme is a direct analog of the Wegner estimate [40] for multi-particle operators. It can be proved under the assumption (W1).

Theorem 5.1. *Let $V : \mathcal{Z} \times \Omega \rightarrow \mathbb{R}$ be a random field satisfying the hypothesis (W1). Then for any $\beta' \in (0, \beta)$ and large L_0 , the following bound holds true:*

$$\forall \mathbf{x} \in \mathcal{Z}^N \sup_{E \in \mathbb{R}} \mathbb{P} \{ \mathbf{B}_L(\mathbf{x}) \text{ is } E\text{-R} \} \leq e^{-L^{\beta'}} \quad (5.1)$$

With $\beta = 1/2$, it suffices to set $\beta' = 1/4$.

Naturally, Theorem 5.1 is based on a probabilistic bound on the events of the form $\{\omega : \text{dist}(E, \Sigma(\mathbf{H}_{\mathbf{B}_L(\mathbf{x})})) < s\}$, with $s = e^{-L^{\beta}}$. An optimal Wegner-type

bound, for an IID random potential V with Lipschitz-continuous marginal PDF F_V , would have the form

$$\mathbb{P} \{ \omega : \text{dist}(E, \Sigma(\mathbf{H}_{\mathbf{B}_L(\mathbf{x})})) < s \} \leq \text{Const} |\mathbf{B}_L(\mathbf{x})| s. \quad (5.2)$$

Indeed, such a (multi-particle) bound was proved by the author and Yuri Suhov for IID random potentials with PDF analytic in a strip $\{E \in \mathbb{C} : |\text{Im } E| < \epsilon\}$, for some $\epsilon > 0$ (hence, having an analytic probability density). This covers the so-called α -stable laws indexed by a real parameter $\alpha \in [1, 2]$ (not to be confused with the scaling exponent α appearing in the recursion $L_{k+1} = [L_k^\alpha]$), where $\alpha = 2$ corresponds to the Gaussian law and $\alpha = 1$ to the Cauchy law. The proof is based on Molchanov's rigorous path integral formula for the unitary propagators e^{-itH} (for finite-difference operators H) and follows essentially the same path as in Carmona's proof of analyticity of the Density of States (DoS) for tight-binding Anderson Hamiltonians; cf. [18]. The same method applies to the two-volume bounds given in Theorem 5.3 below, again for analytic PDF F_V .

Kirsch² [33] proved an optimal multi-particle Wegner estimate of the form (5.2) for uniformly Lipschitz continuous marginal probability distributions. The proof of Theorem 5.1 in [8], for possibly singular (but uniformly continuous) marginal probability distributions is based on Stollmann's lemma from [38] which gives rise to a non-optimal volume dependence; the latter, however, is absorbed in the fractional exponent and is more than sufficient for applications to the multi-particle MSA. Stollmann's lemma itself is very general and not sensitive to the regularity properties of the marginal distribution, but is effective (suitable for MSA applications) only for marginal PDFs featuring at least log-Hölder continuity, with suitable parameters.

A one-volume EVC multi-particle bound for continuous systems (Anderson Hamiltonians with an alloy-type random potential), with optimal volume dependence, has been reported by Hislop and Klopp [31]. A variant of such bound (again, for alloy-type potentials in \mathbb{R}^d) based on Stollmann's lemma, with a non-optimal volume dependence, was proven earlier in [6] and used in [16], in the proof of multi-particle dynamical localization for Anderson-type Hamiltonians in \mathbb{R}^d .

5.2. "Separability" in the multi-particle configuration space. All above mentioned one-volume EVC bounds suffice for the proof of the a.s. exponential decay (hence, square summability) of *multi-particle* Green functions. This point was not clear earlier, since Simon–Wolf method from [37] does not apply to multi-particle Hamiltonians. However, both the variable-energy MPMSA from [10] and the derivation of the VEMPMSA bounds from their fixed-energy counterparts (presented here) require two-volume EVC bounds which are more sophisticated for multi-particle systems than in the conventional, single-particle localization theory. It turns out that the positions of the balls play an important role. For this reason, following [10], we introduce

Definition 5.1. A ball $\mathbf{B}_L^{(N)}(\mathbf{x})$ is separable from a ball $\mathbf{B}_L^{(N)}(\mathbf{y})$ if there exists a decomposition $\mathbf{B}_L^{(N)}(\mathbf{x}) = \mathbf{B}_L^{(n')}(\mathbf{x}') \times \mathbf{B}_L^{(n'')}(\mathbf{x}'')$ such that (cf. Eqn (2.3))

$$\Pi \mathbf{B}_L^{(n')}(\mathbf{x}') \cap \Pi \mathbf{B}_L^{(N)}(\mathbf{y}) = \emptyset. \quad (5.3)$$

A pair $(\mathbf{B}_L^{(N)}(\mathbf{x}), \mathbf{B}_L^{(N)}(\mathbf{y}))$ is separable if one of the balls is separable from the other.

²See also the preprint [arXiv:math-ph/0704.2664](https://arxiv.org/abs/math-ph/0704.2664) (2007).

Pictorially, (5.3) says that there is a sub-sample of the potential which determines the operator $\mathbf{H}_{\mathbf{B}_L(\mathbf{y})}$ completely, while $\mathbf{H}_{\mathbf{B}_L(\mathbf{x})}$ (more precisely, its component relative to a sub-configuration \mathbf{x}') has a non-degenerate (and controllable) probability distribution, conditional on $\mathbf{H}_{\mathbf{B}_L(\mathbf{y})}$.

Lemma 5.2. *For any $\mathbf{x} \in \mathcal{Z}^N$ there is $R(\mathbf{x})$ such that any ball $\mathbf{B}_L^{(N)}(\mathbf{y})$ with $\rho(\mathbf{x}, \mathbf{y}) \geq R(\mathbf{x})$ is separable from $\mathbf{B}_L^{(N)}(\mathbf{x})$.*

Proof. Let $r = \text{diam}(\Pi\mathbf{x})$ and set $R = R(\mathbf{x}) = r + 3NL$. Consider a ball $\mathbf{B}_L^{(N)}(\mathbf{y})$ with $\rho(\mathbf{x}, \mathbf{y}) > R$. By definition of the max-distance, there is at least one particle position $y_{i_0} \in \Pi\mathbf{y}$ such that $d(y_{i_0}, x_{i_0}) \geq R$. Consider the maximal connected component $\Lambda_{\mathbf{y}} := \cup_{i \in \mathcal{J}} \mathbf{B}_{2L}(y_i)$ of the union $\cup_i \mathbf{B}_{2L}(y_i)$ containing y_{i_0} ; its diameter is bounded by $2NL$, and by triangle inequality,

$$\text{dist}(\Lambda_{\mathbf{y}}, \Pi\mathbf{B}_L(\mathbf{x})) \geq R - (\text{diam } \Pi\mathbf{x} + 2L) - \text{diam } \Lambda_{\mathbf{y}} > 0.$$

Taking as the subconfiguration \mathbf{y}' the union $\cup_{i \in \mathcal{J}} \{y_i\}$, we conclude that $\mathbf{B}_L^{(N)}(\mathbf{y})$ is indeed separable from $\mathbf{B}_L^{(N)}(\mathbf{x})$. \square

Theorem 5.3 (Cf. Lemma 2 in [10]). *Under the hypothesis (W1), the following bound holds true for any pair of separable balls $\mathbf{B}_L(\mathbf{x})$, $\mathbf{B}_L(\mathbf{y})$ and all $s \in [0, 1]$:*

$$\mathbb{P} \left\{ \text{dist} \left(\Sigma(\mathbf{H}_{\mathbf{B}_L(\mathbf{x})}), \Sigma(\mathbf{H}_{\mathbf{B}_L(\mathbf{y})}) \right) \leq s \right\} \leq \text{Const } L^{(2N+1)d} s^\delta. \quad (5.4)$$

Lemma 2 in [10] is formally stated for operators on a lattice $(\mathbb{Z}^d)^N$, but its proof, based on a very general Stollmann's lemma on monotone functions [38, 39], applies, with minor notational modifications, also to Hamiltonians on graphs, provided that the extrenal random potential field $V : \mathcal{Z} \times \Omega \rightarrow \mathbb{R}$ is IID with Hölder-continuous marginal distribution.

The dependence of the upper bound on the EF correlators in Theorem 3.1 upon the positions $\hat{\mathbf{x}}_j$, \mathbf{x} is explained by the nature of the two-volume EVC bound in Theorem 5.3: under the hypothesis (W1), "tunneling" between quantum states \mathbf{x} , \mathbf{y} is ruled out with high probability only if the balls $\mathbf{B}_L(\mathbf{x})$, $\mathbf{B}_L(\mathbf{y})$, with a suitable $L = O(\rho(\mathbf{x}, \mathbf{y}))$, are separable.

5.3. "Weak separability" at large distances.

Definition 5.2. A ball $\mathbf{B}_L^{(N)}(\mathbf{x}) \subset \mathcal{Z}^N$ is weakly separable from $\mathbf{B}_L^{(N)}(\mathbf{y})$ if there exists a ball $\Lambda \subset \mathcal{Z}$ in the 1-particle configuration space, of diameter $R \leq 2NL$, and subsets $\mathcal{J}_1, \mathcal{J}_2 \subset [1, N] \cap \mathbb{Z}$ such that $|\mathcal{J}_1| > |\mathcal{J}_2|$ (possibly, $\mathcal{J}_2 = \emptyset$) and

$$\begin{cases} \Pi_{\mathcal{J}_1} \mathbf{B}_L^{(N)}(\mathbf{x}) \cup \Pi_{\mathcal{J}_2} \mathbf{B}_L^{(N)}(\mathbf{y}) \subseteq \Lambda, \\ \Pi_{\mathcal{J}_1^c} \mathbf{B}_L^{(N)}(\mathbf{x}) \cap \Lambda = \emptyset, \\ \Pi_{\mathcal{J}_2^c} \mathbf{B}_L^{(N)}(\mathbf{y}) \cap \Lambda = \emptyset. \end{cases} \quad (5.5)$$

A pair of balls $(\mathbf{B}_L^{(N)}(\mathbf{x}), \mathbf{B}_L^{(N)}(\mathbf{y}))$ is weakly separable if at least one of the balls is weakly separable from the other.

The physical meaning of the weak separability is that in a certain region of the one-particle configuration space, there are more particles from configuration \mathbf{x} than from \mathbf{y} . As a result, some local fluctuations of the random potential $V(\cdot; \omega)$ have a stronger influence on EVs relative to $\mathbf{B}_L(\mathbf{x})$ than on EVs relative to $\mathbf{B}_L(\mathbf{y})$. It is easy to see that this condition is indeed weaker than the separability.

Lemma 5.4 (Cf. Lemma 2.3 in [11]). *If $\rho_S(\mathbf{x}, \mathbf{y}) > 3NL$, then balls $\mathbf{B}_L^{(N)}(\mathbf{x})$ and $\mathbf{B}_L^{(N)}(\mathbf{y})$ are weakly separable.*

The lower bound by $3NL$ is not sharp, but slightly less cumbersome than the one established in [11]; the actual value of the constant (in front of NL) is irrelevant. The proof is given in [11], formally, applies to the case where $\mathcal{Z} = \mathbb{Z}^d$, but the extension to more general graphs is straightforward. The main idea can be summarized as follows: two balls of radius L with centers \mathbf{x}, \mathbf{y} are *not* weakly separable iff all particles from \mathbf{x} occupy approximately the same positions as the particles from \mathbf{y} (up to a permutation: this is why the symmetrized distance ρ_S is used). Specifically, there has to be a permutation $\pi \in \mathfrak{S}_N$ such that $\forall i \in [1, N], d(x_i, y_{\pi(i)}) = O(L)$, yielding $\rho_S(\mathbf{x}, \mathbf{y}) = O(NL)$.

We will call a pair of balls $\mathbf{B}_L^{(N)}(\mathbf{x}), \mathbf{B}_L^{(N)}(\mathbf{y})$ *distant* iff $\rho_S(\mathbf{x}, \mathbf{y}) \geq 3NL$.

Theorem 5.5 (Cf. Lemma 3.1 in [11]). *Under the hypothesis (W2), for any pair of weakly separable (e.g., distant) balls $\mathbf{B}_L(\mathbf{x}), \mathbf{B}_L(\mathbf{y})$ the following bound holds true:*

$$\mathbb{P} \left\{ \text{dist}(\Sigma(\mathbf{H}_{\mathbf{B}_L(\mathbf{x})}), \Sigma(\mathbf{H}_{\mathbf{B}_L(\mathbf{y})})) \leq s \right\} \leq h_L(2s),$$

with $h_L(s)$ defined in (2.15) and constants C, C', A, A', B, B' defined in (W2):

$$h_L(s) = |\mathbf{B}_L(\mathbf{x})| \cdot |\mathbf{B}_L(\mathbf{y})| CL^A s^B + C' L^{A'} s^{B'}. \quad (5.6)$$

The proof given in [11] in the case of a periodic lattice $\mathcal{Z} = \mathbb{Z}^d$ extends with no difficulty to more general graphs.

6. FIXED-ENERGY MULTI-PARTICLE SCALE INDUCTION

It is to be emphasized that the entire fixed-energy scaling procedure, for any number $N \geq 2$ particles, does not reveal the difficulty, mentioned in the Introduction and related to the "structural" resonances. As a result, the exponential decay (hence, square-summability) of Green functions at a fixed energy can be established under a weaker hypothesis of Hölder (or log-Hölder) continuity of the marginal PDF F_V of the random potential field $V : \mathcal{Z} \times \Omega \rightarrow \mathbb{R}$.

6.1. Scaling of Green functions in absence of tunneling.

Definition 6.1. A ball $\mathbf{B}_{L_{k+1}}(\mathbf{u})$ is called *E-tunneling (E-T)* if it contains two disjoint (E, m) -S balls of radius L_k , and *E-non-tunneling (E-NT)*, otherwise.

Lemma 6.1. *If a ball $\mathbf{B}_{L_{k+1}}(\mathbf{u})$ is E-NR and E-NT, then it is (E, m) -NS.*

Proof. See the proof of Lemma 5.1 from [15]. Its statement is deterministic and not specific to the analytic nature of the potential, thus applies to DSO on graphs with single- and multi-particle structure of the potential. \square

6.2. Main inductive bound. Given a number $\kappa > \frac{2\alpha}{2-\alpha}Nd$, pick any $0 < \theta < \frac{2\alpha}{2-\alpha} - \frac{2Nd}{\kappa}$ and introduce the following double sequence:

$$P(N, k) = P(N, k; \hat{N}, \kappa, \theta) = 3^{\hat{N}-N} \kappa (1 + \theta)^k, \quad N = 1, \dots, \hat{N}; \quad k \geq 0, \quad (6.1)$$

and a statement, or property, depending upon integer parameters $N \in [1, \hat{N}], k \geq 1$, and an interval $I \subseteq \mathbb{R}$, which we are going to prove by induction:

$$S(I, N, k): \quad \forall E \in I \quad \forall n \in [1, N] \quad \forall \mathbf{x} \in \mathcal{Z}^n$$

$$\mathbb{P} \left\{ B_{L_k}^{(n)}(\mathbf{x}) \text{ is } (E, m)\text{-S} \right\} \leq L_k^{-3^{\hat{N}-n\kappa}} = L_k^{-P(n,k)}. \quad (6.2)$$

6.3. Initial scale bounds. First, consider the case of strongly disordered systems.

Lemma 6.2. *For any $L_0 \geq 2$, $m \geq 1$, $\hat{N} \geq 1$ and $N \in [1, \hat{N}]$ there is $g^* = g^*(m, p, n, L_0) < \infty$ such that if $|g| \geq g^*$, then the property $S(I, N, 0)$ holds true.*

The proof is similar to its well-known 1-particle counterpart (cf., e.g., [19]) and will be omitted. It also follows easily from the Wegner-type estimate (cf. Theorem 5.1) combined with the Combes–Thomas estimate.

For energies close to the lower edge of the spectrum, an initial scale bound suitable for the purposes of the MPMSA has been proved by Ekanga [21, 22].

In the framework of multi-particle systems with interaction, a third scenario appears where the localization can be proved, merely by an adaptation of the initial scale estimate: systems localized without interaction and perturbed by a sufficiently weak interaction (e.g., of short range). Such scenario has been first analyzed by Aizenman and Warzel [3] with the help of the MPFMM. Ekanga [22] adapted the method of [10] to weakly interacting systems. In particular, he proved that N -particle systems in \mathbb{Z}^1 , with two-parameter Hamiltonian

$$\mathbf{H} = -\Delta + g\mathbf{V}(\mathbf{x}; \omega) + h\mathbf{U}(\mathbf{x})$$

are localized for any nonzero $g \in \mathbb{R}$ and all sufficiently small $|h| \leq h^*(|g|)$. Note, however, that the random potential is required to have log-Hölder continuous marginal PDF F_V (with suitable parameters), so singular (e.g., Bernoulli) potentials are not allowed in [22]. Indeed, the proof follows the MSA scheme (which is perturbative, by its nature), for the specifically one-dimensional methods based on Furstenberg theory do not apply even to 2-particle Hamiltonians. The same difficulty is encountered in [3, 4] where the fractional moment method is also perturbative.

Lemma 6.3 (Cf. [21, 22]). *For any $L_0 \geq 2$, $m \geq 1$, $\hat{N} \geq 1$ and $N \in [1, \hat{N}]$ there is $g^* = g^*(m, p, n, L_0) < \infty$ such that if $|g| \geq g^*$, then the property $S(I, N, 0)$ holds true.*

Proof. (We only sketch the proof.) Owing to the positivity of interaction U , of the external potential V and of the graph Laplacian $-\Delta$, one has

$$\mathbf{H}^{(N)} \geq H^{(1)} \otimes \mathbf{1}^{(N-1)}.$$

Therefore, a lower bound for the spectrum of $\mathbf{H}^{(N)}$ is provided, e.g., by the lowest eigenvalue of the single-particle Hamiltonian. For the latter, it is well-known and is due to the Lifshitz tails asymptotics. \square

The case of the random potential bounded from below by an arbitrary quantity $E_* > -\infty$ can be treated similarly, replacing $V(x; \omega)$ by $V(x; \omega) - E_0$ and $\mathbf{H}^{(N)}$ by $\mathbf{H}^{(N)} - NE_0 \mathbf{1}$. In the case where $E_0 = -\infty$ (e.g., Gaussian potentials) the proof is even easier and follows virtually the same path as for strongly disordered systems.

For brevity, below we treat **only the case of strong disorder**; an adaptation to the band-edge localization merely requires the control of the "mass" $m > 0$ and simple modifications of scaling formulae.

6.4. Green functions in decoupled systems.

Definition 6.2. An N -particle ball $\mathbf{B}_L(\mathbf{u})$ is called partially interactive (PI) if $\text{diam}(\Pi\mathbf{u}) > 11NL$, and fully interactive (FI), otherwise.

If the interaction has finite range r_0 and $L \geq 8r_0$, a polydisk $\mathbf{B}_L(\mathbf{u})$ can be represented as follows:

$$\mathbf{B}_L(\mathbf{u}) = \mathbf{B}_L^{(N)}(\mathbf{u}) = \mathbf{B}_L^{(n')}(\mathbf{u}') \times \mathbf{B}_L^{(n'')}(\mathbf{u}'') \quad (6.3)$$

where $n' + n'' = N$ and the subconfigurations $\mathbf{u} = (\mathbf{u}', \mathbf{u}'')$, $\mathbf{u}' \in \mathcal{X}^{n'}$, $\mathbf{u}'' \in \mathcal{X}^{n''}$ satisfy the condition

$$\rho\left(\Pi\mathbf{B}_L^{(n')}(\mathbf{u}'), \Pi\mathbf{B}_L^{(n'')}(\mathbf{u}'')\right) > r_0.$$

As a result, the operator $\mathbf{H}_{\mathbf{B}_L(\mathbf{u})}^{(N)}$ in $\mathbf{B}_L(\mathbf{u})$ is algebraically decomposable in the following way:

$$\mathbf{H}_{\mathbf{B}_L(\mathbf{u})}^{(N)} = \mathbf{H}_{\mathbf{B}_L(\mathbf{u}')}^{(n')} \otimes \mathbf{1}^{(n'')} + \mathbf{1}^{(n')} \otimes \mathbf{H}_{\mathbf{B}_L(\mathbf{u}'')}^{(n'')} \quad (6.4)$$

since, for any $\mathbf{x} = (\mathbf{x}', \mathbf{x}'') \in \mathbf{B}_L^{(N)}(\mathbf{u})$, the interaction between the sub-configurations \mathbf{x}' and \mathbf{x}'' vanishes. Such a decomposition, if it exists, may be non-unique. We will assume that one such decomposition (referred to as the canonical one) is associated in some way with every N -particle PI ball.

Lemma 6.4. Assume that the interaction \mathbf{U} satisfies the condition (U1). Fix an energy $E \in \mathbb{R}$. Consider a PI N -particle ball with canonical decomposition $\mathbf{B}_{L_k}^{(N)}(\mathbf{u}) = \mathbf{B}_{L_k}^{(n')}(\mathbf{u}') \times \mathbf{B}_{L_k}^{(n'')}(\mathbf{u}'')$ and a sample of the potential V such that

- (a) $\mathbf{B}_{L_k}^{(N)}(\mathbf{u})$ is E -NR.
- (b) $\forall \lambda'' \in \Sigma(\mathbf{H}_{\mathbf{B}_{L_k}^{(n'')}(\mathbf{u}'')}^{(n'')})$ the ball $\mathbf{B}_{L_k}^{(n')}(\mathbf{u}')$ is $(E - \lambda'', m)$ -NS;
- (c) $\forall \lambda' \in \Sigma(\mathbf{H}_{\mathbf{B}_{L_k}^{(n')}(\mathbf{u}')}^{(n')})$ the ball $\mathbf{B}_{L_k}^{(n'')}(\mathbf{u}'')$ is $(E - \lambda', m)$ -NS.

Then the ball $\mathbf{B}_{L_k}^{(N)}(\mathbf{u})$ is (E, m) -NS.

Proof. See subsection 10.1. □

Corollary 6.5. Assume the property $\mathcal{S}(I, N, k)$ and let $L_0 \geq 4C_d^N$. Then for all $E \in I$ and any PI ball of the form $\mathbf{B}_{L_k}^{(N)}(\mathbf{x}) \subset \mathcal{Z}^N$,

$$\mathbb{P}\left\{ \mathbf{B}_{L_k}^{(N)}(\mathbf{x}) \text{ is } (E, m)\text{-S} \right\} \leq \frac{1}{4}C_d^{-2N}L_k^{-\frac{11}{4}P(N,k)} \quad (6.5)$$

Proof. Denote by \mathcal{S}_k the event in the LHS of (6.5). Consider the canonical decomposition $\mathbf{B}_{L_k}^{(N)}(\mathbf{x}) = \mathbf{B}_{L_k}^{(n')}(\mathbf{x}') \times \mathbf{B}_{L_k}^{(n'')}(\mathbf{x}'')$. By virtue of Lemma 6.1,

$$\mathbb{P}\{\mathcal{S}_k\} < \mathbb{P}\left\{ \mathbf{B}_{L_k}^{(N)}(\mathbf{x}) \text{ is } E\text{-R} \right\} + \mathbb{P}\left\{ \mathbf{B}_{L_k}^{(N)}(\mathbf{x}) \text{ is } E\text{-NR and } (E, m)\text{-S} \right\}. \quad (6.6)$$

The first term in the RHS is bounded with the help of Theorem 5.1, so we focus on the second term. Apply Lemma 6.4: since the option (a) is ruled out, it remains to assess the probability of events listed in options (b) and (c). Consider the former:

$$\begin{aligned} \mathbb{P}\{\mathcal{S}'\} &\equiv \mathbb{P}\left\{ \exists \lambda'' \in \Sigma(\mathbf{H}_{\mathbf{B}_{L_k}^{(n'')}(\mathbf{x}'')}^{(n'')}) : \mathbf{B}_{L_k}^{(n')}(\mathbf{x}') \text{ is } (E - \lambda'', m)\text{-S} \right\} \\ &= \mathbb{E}\left[\mathbb{P}\left\{ \exists \lambda'' \in \Sigma(\mathbf{H}_{\mathbf{B}_{L_k}^{(n'')}(\mathbf{x}'')}^{(n'')}) : \mathbf{B}_{L_k}^{(n')}(\mathbf{x}') \text{ is } (E - \lambda'', m)\text{-S} \right\} \middle| \mathfrak{F}'' \right] \end{aligned}$$

where \mathfrak{F}'' is the sigma-algebra generated by the values of the random potential in $\mathbf{B}_{L_k}^{(n'')}(x'')$. By definition of the canonical decomposition of a PI ball,

$$\Pi \mathbf{B}_{L_k}^{(n')}(x') \cap \Pi \mathbf{B}_{L_k}^{(n'')}(x'') = \emptyset, \quad (6.7)$$

and since the random field V is IID, for any $E'' \in \mathbb{R}$, including $E - \lambda''$, one has

$$(\text{a.s.}) \quad \mathbb{P} \left\{ \mathbf{B}_{L_k}^{(n')}(x') \text{ is } (E'', m)\text{-S} \mid \mathfrak{F}'' \right\} = \mathbb{P} \left\{ \mathbf{B}_{L_k}^{(n')}(x') \text{ is } (E'', m)\text{-S} \right\}. \quad (6.8)$$

On the other hand, by the assumed property $\mathbf{S}(I, N-1, k)$, for $n' \leq N-1$,

$$\mathbb{P} \left\{ \mathbf{B}_{L_k}^{(n')}(x') \text{ is } (E'', m)\text{-S} \right\} \leq C_d^{-2N} L_k^{-P(N-1, k)} = C_d^{-2N} L_k^{-3P(N, k)}. \quad (6.9)$$

Therefore, we obtain, with $L_0 \geq 4C_d^N$,

$$\begin{aligned} \mathbb{P} \{ \mathcal{S}' \} &\leq |\mathbf{B}_{L_k}^{(n'')}(x'')| \sup_{E'' \in \mathbb{R}} \mathbb{P} \left\{ \mathbf{B}_{L_k}^{(n')}(x') \text{ is } (E'', m)\text{-S} \right\} \\ &\leq \frac{C_d^N}{C_d^{2N}} L_k^{Nd} L_k^{-3P(N, k)} \leq \frac{1}{4} C_d^{-2N} L_k^{-\frac{11}{4}P(N, k) - (\frac{1}{4}P(N, k) - Nd - 1)} \\ &\leq \frac{1}{4} C_d^{-2N} L_k^{-\frac{11}{4}P(N, k)} \end{aligned} \quad (6.10)$$

since $p > 6Nd$, so $\frac{1}{4}P(N, k) > \frac{3}{2}Nd \geq Nd + 1$ (for $N \geq 2$). Similarly,

$$\mathbb{P} \left\{ \exists \lambda' \in \Sigma(\mathbf{H}_{\mathbf{B}_{L_k}^{(n')}(x')}) : \mathbf{B}_{L_k}^{(n'')}(x'') \text{ is } (E - \lambda', m)\text{-S} \right\} \leq \frac{1}{4} C_d^{-2N} L_k^{-\frac{11}{4}P(N, k)}. \quad (6.11)$$

Collecting (5.1), (6.6), (6.10) and (6.11), the assertion follows. \square

6.5. Scale induction. Introduce the following notations: with $k \geq 0$,

$$\begin{aligned} P_k &= \sup_{\mathbf{x} \in \mathcal{Z}^N} \mathbb{P} \{ \mathbf{B}_{L_k}(\mathbf{x}) \text{ is } (E, m)\text{-S} \}, \\ Q_{k+1} &= 4 \sup_{\mathbf{x} \in \mathcal{Z}^N} \mathbb{P} \{ \mathbf{B}_{L_{k+1}}(\mathbf{x}) \text{ is } E\text{-R} \}, \\ S_{k+1} &= \sup_{\mathbf{x} \in \mathcal{Z}^N} \mathbb{P} \{ \mathbf{B}_{L_k}(\mathbf{x}) \text{ contains a PI } (E, m)\text{-S ball of radius } L_k \}. \end{aligned}$$

For future use, note that by Corollary 6.5, with $\kappa > 6Nd$ and $0 < \theta < 1/6$,

$$\begin{aligned} S_{k+1} &\leq \frac{1}{2C_d^{2N}} L_k^{-\frac{11}{4}P(N, k)} \cdot C_d^N L_{k+1}^{Nd} \leq \frac{1}{4C_d^{2N}} L_{k+1}^{-\frac{11}{4} \cdot \frac{2}{3}P(N, k) - Nd} \\ &\leq \frac{1}{4C_d^{2N}} L_{k+1}^{-\frac{11}{6(1+\theta)}\kappa(1+\theta)^{k+1} - Nd} \leq \frac{1}{4C_d^{2N}} L_{k+1}^{-\frac{11}{6(1+\frac{1}{6})}\kappa(1+\theta)^{k+1} - Nd} \\ &\leq \frac{1}{4C_d^{2N}} L_{k+1}^{-\frac{11}{7}\kappa(1+\theta)^{k+1} - Nd} \leq \frac{1}{4C_d^{2N}} L_{k+1}^{-\kappa(1+\theta)^{k+1}}. \end{aligned} \quad (6.12)$$

Theorem 6.6. Let $\alpha \in (1, 2)$, $\kappa > \frac{2Nd\alpha}{2-\alpha}$ and $\theta \in (0, \frac{2-\alpha}{\alpha} - \frac{2Nd}{\kappa})$; assume the condition (W1). If there is an integer $L_0 \geq 4C_d^N$ such that $\mathbf{S}(I, N, 0)$ is fulfilled and $Q_0 \leq C_d^{-2N} L_0^{-\kappa}$, then $\mathbf{S}(I, N, k)$ holds true for all $k \geq 0$.

Proof. It suffices to derive $\mathbf{S}(I, N, k+1)$ from $\mathbf{S}(I, N, k)$, so assume the latter. By virtue of Lemma 6.1, if a ball $\mathbf{B}_{L_{k+1}}(u)$ is (E, m) -S, then it is either E -R or E -T.

By Eqn (6.12), the probability to have at least one PI (E, m) -S ball of radius L_k inside $B_{L_{k+1}}(u)$ obeys $S_{k+1} \leq \frac{1}{4C_d^{2N}} L_{k+1}^{-\kappa(1+\theta)^{k+1}}$.

Further, there are $< \frac{1}{2} C_d^{2N} L_{k+1}^{2Nd}$ pairs of disjoint L_k -balls in $B_{L_{k+1}}(u)$, thus

$$P_{k+1} \leq \frac{1}{2} C_d^{2N} L_{k+1}^{2Nd} P_k^2 + \frac{1}{4} Q_{k+1} + S_{k+1}.$$

By Theorem 5.1, $Q_{k+1} \leq \text{Const } L_{k+1}^{Nd} e^{-L_{k+1}^\beta}$. The function

$$f : L \mapsto \ln(\text{const } L^{-\kappa}) - \ln(\text{const } L^d e^{-L^\beta}) = L^\beta - \text{const } \ln L$$

on $[1, +\infty)$ is either non-negative or admits a unique zero. In either case, the assumption $Q_0 \leq C_d^{-2N} L_0^{-\kappa}$ implies $Q_{k+1} \leq C_d^{-2N} L_{k+1}^{-\kappa}$ for all $k \geq 0$, so

$$\frac{1}{4} Q_{k+1} + S_{k+1} \leq \frac{1}{2} C_d^{-2N} L_{k+1}^{-\kappa(1+\theta)^{k+1}}.$$

Therefore,

$$\begin{aligned} P_{k+1} &\leq \frac{1}{2} C_d^{2N} L_{k+1}^{2Nd} P_k^2 + \frac{1}{2} C_d^{-2N} L_{k+1}^{-\kappa(1+\theta)^{k+1}} \leq \frac{C_d^2 N}{C_d^{4N}} L_{k+1}^{-\frac{2\kappa(1+\theta)^k}{\alpha} + 2Nd} \\ &\leq C_d^{-2N} L_{k+1}^{-\kappa(1+\theta)^{k+1}}, \end{aligned}$$

provided that $\frac{2\kappa}{\alpha} + 2Nd \geq \kappa$ (i.e., $\kappa \geq \frac{2Nd\alpha}{2-\alpha}$) and $\theta = \left(\frac{2}{\alpha} - \frac{2d}{\kappa}\right) - 1 > 0$. \square

This marks the end of the fixed-energy multi-particle multi-scale analysis.

7. FROM FIXED TO VARIABLE ENERGY: FIRST APPROACH

Now we establish a fairly general relation between fixed-energy probabilistic estimates on the Green functions and variable-energy bounds for two disjoint finite volumes. It does not matter how the probabilistic input is obtained; in particular, the results of this section can be combined both with the MSA, performed for each fixed energy E in a given interval $I \subset \mathbb{R}$, and with the FMM (which always starts as a fixed-energy analysis).

7.1. Derivation of the first variable-energy bound. The results of this subsection³ are based on a straightforward adaptation of the techniques developed by Elgart *et al.* [23]. In fact, the assertion of Theorem 7.1 below is a mere encapsulation of an argument from [23] in a statement involving four parameters which can be adapted in various ways to particular models.

It is convenient to assume that $|I| = 1$, so the interval I with the Lebesgue measure $\text{mes}(\cdot)$ is a probability space, and so is the product space $(\Omega \times I, \mathbb{P} \times \text{mes})$. Given $L \in \mathbb{N}$ and points $\mathbf{x}, \mathbf{y} \in \mathcal{Z}^N$, set for brevity

$$\mathbf{F}_{\mathbf{x}, \mathbf{y}}(E) = |\mathbf{G}_{\mathbf{B}_L}(\mathbf{x}, \mathbf{y}; E)|, \quad \mathbf{F}_{\mathbf{x}}(E) = \max_{\mathbf{y} \in \partial^{\mathbf{B}_L}(\mathbf{x})} \mathbf{F}_{\mathbf{x}, \mathbf{y}}(E), \quad (7.1)$$

and introduce the subsets of I parameterized by $a > 0$:

$$\mathcal{E}_{\mathbf{x}, \mathbf{y}}(a) = \{E \in I : \mathbf{F}_{\mathbf{x}, \mathbf{y}}(E) \geq a\}, \quad \mathcal{E}_{\mathbf{x}}(a) = \{E \in I : \mathbf{F}_{\mathbf{x}}(E) \geq a\}. \quad (7.2)$$

(The L -dependence will be often omitted for brevity.)

³I thank Tom Spencer and Sasha Sodin for stimulating discussions of the works [23, 24], and Ivan Veselić for a fruitful exchange on this subject.

Theorem 7.1. *Let $L \geq 0$, $\mathbf{x} \in \mathcal{Z}^N$, $\mathbf{y} \in \partial^- \mathbf{B}_L(\mathbf{x})$. Let $\{\lambda_j\}_{j=1}^N$ be the eigenvalues of the operator $\mathbf{H}_{\mathbf{B}_L(\mathbf{x})}(\omega)$ and $I \subset \mathbb{R}$ an interval of unit length. Let be given numbers $a, b, c, \mathcal{P}_L > 0$ such that*

$$b \leq \min \{ |\mathbf{B}_L(\mathbf{x})|^{-1} a c^2, c \}, \quad (7.3)$$

and for all $E \in I$

$$\mathbb{P} \{ \mathcal{E}_{\mathbf{x}}(a) \} \equiv \mathbb{P} \{ \mathbf{F}_{\mathbf{x}}(E) \geq a \} \leq \mathcal{P}_L. \quad (7.4)$$

There is an event $\mathcal{B}_{\mathbf{x}}(b)$ with $\mathbb{P} \{ \mathcal{B}_{\mathbf{x}}(b) \} \leq b^{-1} \mathcal{P}_L$ such that $\forall \omega \notin \mathcal{B}_{\mathbf{x}}(b)$, the set

$$\mathcal{E}_{\mathbf{x}}(2a) = \mathcal{E}_{\mathbf{x}}(2a; \omega) = \left\{ E : \mathbf{F}_{\mathbf{x}}(E) \geq 2a \right\}$$

is contained in a union of intervals $\cup_{j=1}^N I_j$, $I_j := \{E : |E - \lambda_j| \leq c\}$, $\lambda_j \in I$.

Proof. Consider the following events parameterized by $b > 0$:

$$\mathcal{B}_{\mathbf{x}}(b) = \{ \omega \in \Omega : \text{mes}(\mathcal{E}_{\mathbf{x}}(a)) > b \}. \quad (7.5)$$

Apply Chebyshev's inequality and the Fubini theorem combined with (7.4):

$$\begin{aligned} \mathbb{P} \{ \mathcal{B}_{\mathbf{x}}(b) \} &\leq b^{-1} \mathbb{E} [\text{mes}(\mathcal{E}_{\mathbf{x}}(a))] \\ &= b^{-1} \int_I dE \mathbb{E} [\mathbf{1}_{\{ \mathbf{F}_{\mathbf{x}}(E) \geq a \}}] \leq b^{-1} \mathcal{P}_L. \end{aligned} \quad (7.6)$$

Now fix any $\omega \notin \mathcal{B}_{\mathbf{x}}(b)$, so that $\text{mes}(\mathcal{E}_{\mathbf{x}}(a); \omega) \leq b$. There is a subset $\{\lambda_j\}_{j=1}^{N'}$ of the EVs of the operator $\mathbf{H}_{\mathbf{B}_L(\mathbf{x})}$ such that the Green function $E \mapsto \mathbf{G}_{\mathbf{B}_L(\mathbf{x})}(\mathbf{x}, \mathbf{y}; E)$ reads as a rational function (below we remove the vanishing terms, if any)

$$f : E \mapsto \mathbf{G}_{\mathbf{B}_L(\mathbf{x})}(\mathbf{x}, \mathbf{y}; E) =: \sum_{j=1}^{N'} \frac{\kappa_j}{\lambda_j - E}, \quad N' \leq N := |\mathbf{B}_L(\mathbf{x})|; \quad (7.7)$$

here $\kappa_j = \kappa_j(\mathbf{x}, \mathbf{y}) \neq 0$ and $\sum_j |\kappa_j| \leq \sum_i |\psi_i(\mathbf{x}) \psi_i(\mathbf{y})| \leq N$. Let

$$\mathcal{R}(2c) = \left\{ \lambda \in \mathbb{R} : \min_j |\lambda_j - \lambda| \geq 2c \right\},$$

$$\mathcal{R}(c) = \left\{ \lambda \in \mathbb{R} : \min_j |\lambda_j - \lambda| \geq c \right\}, \quad c > 0.$$

Observe that, with $0 < b \leq c$, $\mathcal{A}_b := \{E : \text{dist}(E, \mathcal{R}(2c)) < b\} \subset \mathcal{R}(c)$, hence, the set \mathcal{A}_b is a union of open sub-intervals at distance $\geq c$ from the spectrum, and on each sub-interval one has $|f'(E)| \leq N c^{-2}$. Let us show by contraposition that, with $\omega \notin \mathcal{B}_{\mathbf{x}, \mathbf{y}}(b)$,

$$\{E : |\mathbf{G}_{\mathbf{B}_L}(\mathbf{x}, \mathbf{y}; E)| \geq 2a\} \cap \mathcal{R}(c) = \emptyset.$$

Assume otherwise, pick any point λ^* in the non-empty set in the LHS, and let $J := \{E' : |E' - \lambda^*| \leq b\} \subset \mathcal{A}_b \subset \mathcal{R}(c)$. Then for any $E \in J$ one has, by (7.3),

$$|f(E)| \geq |f(\lambda^*)| - |J| \sup_{E' \in J} |f'(E')| > 2a - N c^{-2} \cdot b \geq a,$$

so $J \subset \mathcal{E}_{\mathbf{x}, \mathbf{y}}(a)$ and $\text{mes}(\mathcal{E}_{\mathbf{x}, \mathbf{y}}(a)) \geq \text{mes}(J) = 2b > b$, contrary to the choice of ω . Since the set $\mathcal{R}(c)$ is independent of \mathbf{y} , the assertion follows from (7.6). \square

Taking into account the FEMSA bound obtained in Section 6, $\mathcal{P}_{L_k} = \mathcal{P}_k \leq L_k^{-\kappa(1+\theta)^k}$, one can set, e.g., with $\alpha = \frac{3}{2}$ and $\kappa > 6Nd$, for $L \in \{L_k, k \geq 0\}$:

$$a(L_k) = L_k^{-\frac{3\kappa}{5}(1+\theta)^k}, \quad b(L_k) = L_k^{-\frac{\kappa}{5}(1+\theta)^k}, \quad c(L_k) = L_k^{-\left(\frac{\kappa}{5} - \frac{d}{2}\right)(1+\theta)^k}. \quad (7.8)$$

Since $\kappa > 6Nd$, one has $c(L_k) < L_k^{-\frac{\kappa}{30}(1+\theta)^k}$.

Theorem 7.2. *If the property (W1) holds true, then for any pair of separable balls $\mathbf{B}_{L_k}(\mathbf{x})$, $\mathbf{B}_{L_k}(\mathbf{y})$, $k \geq 0$, and a bounded interval $I \subset \mathbb{R}$, one has*

$$\mathbb{P} \left\{ \exists E \in I : \min\{\mathbf{F}_{\mathbf{x}}(E), \mathbf{F}_{\mathbf{y}}(E)\} \geq L_k^{-\frac{3\kappa}{5}(1+\theta)^k} \right\} \leq C L_k^{-\frac{\delta\kappa}{30}(1+\theta)^k + (2N+1)d}.$$

Proof. Introduce the event $\mathcal{B}_b(\mathbf{x})$, $b = b(L_k)$, relative to the ball $\mathbf{B}_{L_k}(\mathbf{x})$ and defined as in Theorem 7.1; similarly, define the event $\mathcal{B}_b(\mathbf{y})$ relative to the ball $\mathbf{B}_{L_k}(\mathbf{y})$ and set $\mathcal{B}_b = \mathcal{B}_b(\mathbf{x}) \cup \mathcal{B}_b(\mathbf{y})$. Further, let $\mathcal{E}_{\mathbf{x}} = \{E \in I : \mathbf{F}_{\mathbf{x}} \geq 2a(L_k)\}$, $\mathcal{E}_{\mathbf{y}} = \{E \in I : \mathbf{F}_{\mathbf{y}} \geq 2a(L_k)\}$, as in Theorem 7.1. For any $\omega \notin \mathcal{B}_b$,

$$\text{dist} \left(\mathcal{E}_{\mathbf{x}}, \Sigma(\mathbf{H}_{\mathbf{B}_{L_k}(\mathbf{x})}) \right) \leq 2c(L_k), \quad \text{dist} \left(\mathcal{E}_{\mathbf{y}}, \Sigma(\mathbf{H}_{\mathbf{B}_{L_k}(\mathbf{y})}) \right) \leq 2c(L_k).$$

Therefore, applying Theorem 5.3, we obtain:

$$\begin{aligned} \mathbb{P} \{ \mathcal{E}_{\mathbf{x}} \cap \mathcal{E}_{\mathbf{y}} \neq \emptyset \} &= \mathbb{P} \{ \text{dist}(\mathcal{E}_{\mathbf{x}}, \mathcal{E}_{\mathbf{y}}) = 0 \} \\ &\leq \mathbb{P} \{ \mathcal{B}_b \} + \mathbb{P} \left\{ \text{dist} \left(\Sigma(\mathbf{H}_{\mathbf{B}_{L_k}(\mathbf{x})}), \Sigma(\mathbf{H}_{\mathbf{B}_{L_k}(\mathbf{y})}) \right) \leq 4c(L_k) \right\} \\ &\leq L_k^{-\frac{4\kappa}{5}(1+\theta)^k} + \text{Const } L_k^{(2N+1)d} (c(L_k))^\delta \\ &\leq \text{Const } L_k^{-\frac{\delta\kappa}{30}(1+\theta)^k + (2N+1)d}. \quad \square \end{aligned} \tag{7.9}$$

7.2. Spectral localization. The assertion of Theorem 7.2 has a structure similar to that of the MSA bound from the work by von Dreifus and Klein [19]. More precisely, it guarantees a decay rate of Green functions slower than exponential, but faster than any power-law. The main difference is that one can rule out, with high probability, only the pairs of singular balls which are separable (and not just disjoint). It is not difficult to adapt the well-known argument from [19] and prove that with probability one, all polynomially bounded solutions to the equation $H(\omega)\psi = E\psi$ are in fact square-summable. The latter property requires a Shnol–Simon type result on spectrally a.e. polynomial boundedness of generalized eigenfunctions. It will follow independently by RAGE (Ruelle–Amrein–Georgescu–Enss) theorems (see a detailed discussion along with a bibliography, e.g., in [17]) from the dynamical localization proven in Section 9.

8. FROM FIXED TO VARIABLE ENERGY: SECOND APPROACH

8.1. The spectral reduction. The next result is a notational adaptation of Theorem 7.1 from [15], formulated there for single-particle Hamiltonians.

Theorem 8.1. *Let be given a ball $\mathbf{B}_L(\mathbf{x})$, $L \geq 1$, and numbers $a(L)$, $b(L)$, $c(L)$, $\mathcal{P}_L > 0$ obeying (7.3) and such that, for some interval I , all $E \in I$,*

$$\mathbb{P} \{ \mathbf{F}_{\mathbf{x}}(E) \geq a \} \leq \mathcal{P}_L. \tag{8.1}$$

Set $K = |\mathbf{B}_L(\mathbf{x})|$. Then the following properties hold true:

(A) *For any $b \geq \mathcal{P}_L$ there exists an event \mathcal{B}_b such that $\mathbb{P} \{ \mathcal{B}_b \} \leq b^{-1}\mathcal{P}_L$ and for any $\omega \notin \mathcal{B}_b$ the set of energies*

$$\mathcal{E}_{\mathbf{x}}(a) = \mathcal{E}_{\mathbf{x}}(a; \omega) = \{ \mathbf{F}_{\mathbf{x}}(E) \geq a \} \cap I$$

is covered by $\tilde{K} < 3K^2$ intervals $J_i = [E_i^-, E_i^+]$, of total length $\sum_i |J_i| \leq b$.

- (B) The endpoints E_i^\pm are determined by the functions $E \mapsto \langle \mathbf{1}_x | (\mathbf{H}_{\mathbf{B}_L(\mathbf{u})} - E)^{-1} | \mathbf{1}_y \rangle$ in such a way that, for the one-parameter family $\mathbf{A}(t) := \mathbf{H}_{\mathbf{B}_L(\mathbf{u})} + t\mathbf{1}$, the endpoints $E_i^\pm(t)$ for the operators $\mathbf{A}(t)$ (replacing $\mathbf{H}_{\mathbf{B}_L(\mathbf{u})}$) have the form

$$E_i^\pm(t) = E_i^\pm(0) + t, \quad t \in \mathbb{R}.$$

Proof. (A) Fix a point $\mathbf{y} \in \partial^- \mathbf{B}_L(\mathbf{x})$ and consider the rational function

$$f_{\mathbf{y}} : E \mapsto \sum_{i=1}^K \frac{\kappa_i}{\lambda_i - E} := \sum_{i=1}^K \frac{\psi_i(x) \psi_i(\mathbf{y})}{\lambda_i - E}.$$

Its derivative has the form

$$f'_{\mathbf{y}}(E) = \sum_{i=1}^K \frac{-\kappa_i}{(\lambda_i - E)^2} =: \frac{\mathcal{P}(E)}{\mathcal{Q}(E)}, \quad \deg \mathcal{P} \leq 2K - 2,$$

and has $\leq 2K - 2$ zeros and $\leq K$ poles, so $f_{\mathbf{y}}$ has $\leq 3K - 1$ intervals of monotonicity $I_{i,\mathbf{y}}$, and the total number of monotonicity intervals of all functions $\{f_{\mathbf{y}}, \mathbf{y} \in \partial^- \mathbf{B}_L(\mathbf{x})\}$ is bounded by $\tilde{K} \leq |\partial^- \mathbf{B}_L(\mathbf{x})|(3K - 1) \leq |\mathbf{B}_L(\mathbf{x})|(3K - 1) < 3K^2$, so

$$\cup_{\mathbf{y} \in \partial^- \mathbf{B}_L(\mathbf{x})} \{E : f_{\mathbf{y}}(E) \geq a\} = \cup_{i=1}^K J_i, \quad J_i = [E_i^-, E_i^+] \subset I,$$

where, obviously, $\sum_i |J_i| \leq \text{mes} \{E : \mathbf{F}_{\mathbf{x}}(E) \geq a\}$.

(B) Consider a one-parameter operator family $\mathbf{A}(t) = \mathbf{H}_{\mathbf{B}_L(\mathbf{u})}(\omega) + t\mathbf{1}$. All these operators share common eigenvectors; the latter determine the coefficients κ_i , so one can choose eigenfunctions $\psi_i(t)$ constant in t and obtain $\kappa_i(t) = \kappa_i(0)$. The eigenvalues of operators $\mathbf{A}(t)$ have the form $\lambda_i(t) = \lambda_i(0) + t$. We conclude that the Green functions, with fixed \mathbf{x} and \mathbf{y} , have the form $f_{\mathbf{x},\mathbf{y}}(E; t) = f_{\mathbf{x},\mathbf{y}}(E - t; 0)$, so that the intervals $J_i(t)$ have indeed the form $J_i(t) = [E_i^- + t, E_i^+ + t]$. \square

Theorem 8.2. Assume the property (W2). Let be given numbers $a > 0$ and $\mathcal{P}_L \in [0, 1]$ such that (8.1) holds true for all $\mathbf{x} \in \mathcal{Z}^N$, then for any pair of weakly separable (e.g., distant) balls $\mathbf{B}_L(\mathbf{x})$, $\mathbf{B}_L(\mathbf{y})$ and for any $b \geq \mathcal{P}_L$, one has

$$\mathbb{P} \{ \exists E \in I : \min\{\mathbf{F}_{\mathbf{x}}(E), \mathbf{F}_{\mathbf{y}}(E)\} \geq a \} \leq 2b^{-1}\mathcal{P}_L + \tilde{h}_L(4b), \quad (8.2)$$

where

$$\tilde{h}_L(s) := K^2 CL^A s^B + C' L^{A'} s^{B'}, \quad K = \max\{|\mathbf{B}_L(\mathbf{x})|, |\mathbf{B}_L(\mathbf{y})|\}.$$

Proof. Fix $b > 0$ (clearly, $b < \mathcal{P}_L$ gives rise to a trivial bound $\mathbb{P}\{\cdot\} < 1$). Define the event $\mathcal{B}_b(\mathbf{x})$ relative to the ball $\mathbf{B}_L(\mathbf{x})$ as in Theorem 8.1. Similarly, define the event $\mathcal{B}_b(\mathbf{y})$ relative to the ball $\mathbf{B}_L(\mathbf{y})$, and let $\mathcal{B}_b = \mathcal{B}_b(\mathbf{x}) \cup \mathcal{B}_b(\mathbf{y})$. Further, let $\mathcal{S}_{\mathbf{x},\mathbf{y}}$ be the event figuring in the LHS of (8.2) and note that

$$\mathbb{P} \{ \mathcal{S}_{\mathbf{x},\mathbf{y}} \} \leq \mathbb{P} \{ \mathcal{B}_b \} + \mathbb{P} \{ \mathcal{S}_{\mathbf{x},\mathbf{y}} \cap \mathcal{B}_b^c \} \leq 2b^{-1}\mathcal{P}_L + \mathbb{P} \{ \mathcal{S}_{\mathbf{x},\mathbf{y}} \cap \mathcal{B}_b^c \}.$$

It remains to assess $\mathbb{P} \{ \mathcal{S}_{\mathbf{x},\mathbf{y}} \cap \mathcal{B}_b^c \}$. By Definition 5.2, there is a ball $\Lambda \subset \mathcal{Z}$ of diameter $R \leq 2NL$ and index subsets $\mathcal{J}_1, \mathcal{J}_2 \subset [1, N]$ such that $n_1 := |\mathcal{J}_1| > |\mathcal{J}_2| =: n_2$ and

$$\begin{cases} \Pi_{\mathcal{J}_1} \mathbf{B}_L^{(N)}(\mathbf{x}) \cup \Pi_{\mathcal{J}_2} \mathbf{B}_L^{(N)}(\mathbf{y}) \subseteq \Lambda, \\ \Pi_{\mathcal{J}_1^c} \mathbf{B}_L^{(N)}(\mathbf{x}) \cap \Lambda = \emptyset, \\ \Pi_{\mathcal{J}_2^c} \mathbf{B}_L^{(N)}(\mathbf{y}) \cap \Lambda = \emptyset. \end{cases} \quad (8.3)$$

Consider the random variables

$$\xi_\Lambda(\omega) := |\Lambda|^{-1} \sum_{z \in \Lambda} V(z; \omega), \quad \eta_z(\omega) := V(z; \omega) - \xi_\Lambda(\omega), \quad z \in \Lambda.$$

Denote the σ -algebra generated by the random variables $\{\eta_z, z \in \Lambda; V(u, \cdot), u \notin \Lambda\}$ by $\mathfrak{F}_{\eta, \Lambda}$. Consider the conditional probability distribution function

$$F_{\xi_\Lambda}(t | \mathfrak{F}_{\eta, \Lambda}) = \mathbb{P} \{ \xi_\Lambda \leq t | \mathfrak{F}_{\eta, \Lambda} \}$$

and its continuity modulus

$$\nu_{\xi_\Lambda}(s | \mathfrak{F}_{\eta, \Lambda}) = \sup_{t \in \mathbb{R}} \text{ess sup} (F_{\xi_\Lambda}(t + s | \mathfrak{F}_{\eta, \Lambda}) - F_{\xi_\Lambda}(t | \mathfrak{F}_{\eta, \Lambda})).$$

Owing to the assumption (W2), for some $C, C', A, A', B, B' \in (0, +\infty)$ we have

$$\forall s \in [0, 1] \quad \mathbb{P} \{ \nu_{\xi_\Lambda}(s | \mathfrak{F}_{\eta, \Lambda}) > CL^A s^B \} \leq C' L^{A'} s^{B'}. \quad (8.4)$$

Using the representation $V(z; \omega) = \xi_\Lambda(\omega) \mathbf{1} + \eta_z(\omega)$ in the set Λ , introduce the respective operator decomposition

$$\mathbf{H}_{B_L(\mathbf{x})}(\omega) = n_1 \xi_\Lambda(\omega) \mathbf{1} + \mathbf{A}_\mathbf{x}(\omega), \quad (8.5)$$

where the operator $\mathbf{A}_\mathbf{x}(\omega)$ is non-random, conditional on $\mathfrak{F}_{\eta, \Lambda}$. For any $\omega \notin \mathcal{B}_b(\mathbf{x})$, the energies E where $\mathbf{F}_\mathbf{x}(E) \geq a$ are covered by intervals $J_{i, \mathbf{x}}$, with $\sum_i |J_{i, \mathbf{x}}| \leq b$. Combining (8.5) and the assertion (B) of Theorem 8.1, we can write

$$J_{i, \mathbf{x}}(\omega) = [\tilde{E}_{i, \mathbf{x}}^-(\omega) + n_1 \xi_\Lambda(\omega), \tilde{E}_{i, \mathbf{x}}^+(\omega) + n_1 \xi_\Lambda(\omega)]$$

where $\tilde{E}_{i, \mathbf{x}}^\pm(\omega)$ are $\mathfrak{F}_{\eta, \Lambda}$ -measurable. For any $\omega \notin \mathcal{B}_b(\mathbf{y})$, the energies E where $\mathbf{F}_\mathbf{y}(E) \geq a$ are covered by intervals $J_{i, \mathbf{y}}$, also obeying $\sum_i |J_{i, \mathbf{y}}| \leq b$. As above,

$$J_{i, \mathbf{y}}(\omega) = [\tilde{E}_{i, \mathbf{y}}^-(\omega) + n_2 \xi_\Lambda(\omega), \tilde{E}_{i, \mathbf{y}}^+(\omega) + n_2 \xi_\Lambda(\omega)]$$

where $\tilde{E}_{i, \mathbf{y}}^\pm(\omega)$ are also $\mathfrak{F}_{\eta, \Lambda}$ -measurable.

Let $\mathcal{B}_b = \mathcal{B}_b(\mathbf{x}) \cup \mathcal{B}_b(\mathbf{y})$ and set $\epsilon_{i, \mathbf{x}} = |J_{i, \mathbf{x}}|$, $\epsilon_{j, \mathbf{y}} = |J_{j, \mathbf{y}}|$. For $\omega \in \mathcal{B}_b^c$, one has $0 \leq \epsilon_{i, \mathbf{x}}, \epsilon_{i, \mathbf{y}} \leq 2b$, so

$$\begin{aligned} \{ \omega : J_{i, \mathbf{x}} \cap J_{j, \mathbf{y}} \neq \emptyset \} \cap \mathcal{B}_b^c &\subset \left\{ |\tilde{E}_{i, \mathbf{x}}^-(\omega) - \tilde{E}_{j, \mathbf{y}}^-(\omega)| \leq \epsilon_{i, \mathbf{x}} + \epsilon_{j, \mathbf{y}} \right\} \cap \mathcal{B}_b^c \\ &\subset \left\{ |(n_1 - n_2) \xi_\Lambda(\omega) - \tilde{E}_{i, j}(\omega)| \leq 2b + 2b \right\} \end{aligned}$$

where $\tilde{E}_{i, j}(\omega)$ is $\mathfrak{F}_{\eta, \Lambda}$ -measurable. Set $\tilde{n} = n_1 - n_2 (\geq 1)$. By (8.4),

$$\begin{aligned} \mathbb{P} \left\{ |\tilde{n} \xi_\Lambda(\omega) - \tilde{E}_{i, j}(\omega)| \leq 4b \right\} &= \mathbb{E} \left[\mathbb{P} \left\{ |\tilde{n} \xi_\Lambda(\omega) - \tilde{E}_{i, j}(\omega)| \leq 4b | \mathfrak{F}_{\eta, \Lambda} \right\} \right] \\ &\leq \mathbb{P} \left\{ \nu_{\xi_\Lambda}(4b | \mathfrak{F}_{\eta, \Lambda}) > CL^A (4b)^B \right\} + CL^A (4b)^B. \end{aligned}$$

Observe that the first probability in the RHS does not depend upon i and j , so it suffices to count it only once. Taking the sum over (i, j) , we obtain the required bound:

$$\begin{aligned} \mathbb{P} \{ \mathcal{S}_{\mathbf{x}, \mathbf{y}} \cap \mathcal{B}^c \} &\leq 2b^{-1} \mathcal{P}_L + \mathbb{P} \{ \omega : \cup_{i, j} (J_{i, \mathbf{x}} \cap J_{j, \mathbf{y}}) \neq \emptyset \} \\ &\leq 2b^{-1} \mathcal{P}_L + \tilde{h}_L(4b). \quad \square \end{aligned}$$

In particular, taking into account Theorem 6.6, we can set, for $L = L_k$,

$$a = a(L_k) = e^{-\gamma(m, L_k)L_k}, \quad \mathcal{P}_{L_k} = L_k^{-\kappa(1+\theta)^k}, \quad b = b(L_k) = L_k^{-\frac{\kappa}{2}(1+\theta)^k}.$$

These settings give rise to the following corollary of Theorem 8.2:

Theorem 8.3. *Assume the property (W2). If there is an integer $L_0 \geq 1$ and numbers $m \geq 1$, $\alpha \in (1, 2)$ such that*

$$\min\{\mathcal{P}_0, \mathcal{Q}_0\} \leq C_d^{-2N} L_0^{-\kappa}, \quad \kappa > \frac{2\alpha Nd}{2-\alpha},$$

then for some $C, a, c > 0$ and all $k \geq 0$, any interval $I \subset \mathbb{R}$ with $|I| \leq 1$ and any pair of distant balls $\mathbf{B}_{L_k}(\mathbf{x}), \mathbf{B}_{L_k}(\mathbf{y})$, the following bound holds true:

$$\mathbb{P}\{E \in I : \mathbf{B}_{L_k}(\mathbf{x}) \text{ and } \mathbf{B}_{L_k}(\mathbf{y}) \text{ are } (E, m)\text{-S}\} \leq C e^{-a \ln^{1+c} L_k}. \quad (8.6)$$

8.2. On the validity of the assumption (8.4). First of all, recall that, by an elementary result on Gaussian distributions, if $V : \mathcal{Z} \times \Omega \rightarrow \mathbb{R}$ is an IID Gaussian field, say, with zero mean and unit variance, the sample average ξ_x of the sample $\{V(z; \omega), z \in \mathbf{B}_L(x)\}$ is independent of the sigma-algebra generated by the "fluctuations" $\eta_z(\omega)$; moreover, it has Gaussian distribution $\mathcal{N}(0, |\mathbf{B}_L(x)|)$ and admits a probability density with $\|p_{\xi_x}\|_\infty \leq \frac{1}{\sqrt{2\pi}} |\mathbf{B}_L(x)|^{1/2}$. In this particular case, Eqn (8.4) can be replaced by a stronger, deterministic bound: the conditional continuity modulus $\nu_{\xi_x}(s | \mathfrak{F}_x)$ is actually independent of the condition and is a.s. bounded by $\|p_{\xi_x}\|_\infty \cdot s$.

Such a situation is rather exceptional, as shows the example of two IID random variables $V_1(\omega), V_2(\omega)$ with uniform distribution $\text{Unif}([0, 1])$. Indeed, in this case $\xi := (V_1 + V_2)/2$, $\eta = (V_1 - V_2)/2$ and the distribution of ξ conditional on η is uniform on the interval I_η of length $O(1 - |\eta|)$, hence, with constant density $O(|1 - |\eta||^{-1})$, for $|\eta| < 1$; for $\eta = \pm 1$, this distribution is concentrated on a single point. However, this example shows also how such a difficulty can be bypassed: excessively "singular" conditional distributions of the sample mean ξ occur only for a set of conditions having a small probability. Using this simple idea, Gaume [27], in the framework of his PhD project, established the property (8.4) for IID random fields with piecewise constant marginal *probability density*. By standard approximation arguments, it can be easily extended to piecewise Lipschitz (or Hölder) continuous *densities*, which is sufficient for most physically relevant applications. We believe that some variant of the property (8.4), perhaps weaker but still sufficient for the purposes of the MSA, holds true in a larger class of IID random fields.

8.3. Exponential spectral localization. The assertion of Theorem 8.3 has the same form as in the conventional MSA bound going back to the work by von Dreifus and Klein [19] (actually, even slightly stronger); therefore, the same argument as in [19] (having its roots in [26]) applies and proves that with probability one, all polynomially bounded solutions to the equation $H(\omega)\psi = E\psi$ are in fact decaying exponentially fast at infinity, thus the operator $H(\omega)$ has a.s. pure point spectrum. The latter property requires a Shnol–Simon type result on spectrally a.e. polynomial boundedness of generalized eigenfunctions; it will also follow by RAGE theorems from the dynamical localization proven in Section 9.

9. FROM MSA TO THE STRONG DYNAMICAL LOCALIZATION

This section is a straightforward adaptation of Section 8 from [15]. Formulated in the most general form, the results of this section apply indifferently to single- and multi-particle Hamiltonians, and the particularity of the latter resides in the geometry of pairs of finite balls involved.

Recall that the first rigorous derivations of the dynamical localization from MSA-type probabilistic bounds on the Green functions have been obtained by Germinet–De Bièvre [28] and Damanik–Stollmann [20]. Germinet and Klein [29] proposed a shorter proof, which can be further simplified in the context of finite-volume operators (which merely should have compact resolvent), and essentially reduced to an elementary application of the Bessel inequality.

In applications to N -particle systems on a graph obeying the growth condition (2.1), the parameter D below has to be replaced by Nd .

9.1. EF correlators in finite balls. Given an interval $I \subset \mathbb{R}$, denote by $\mathcal{B}_1(I)$ the set of all Borel functions $\phi : \mathbb{R} \rightarrow \mathbb{C}$ with $\text{supp } \phi \subset I$ and $\|\phi\|_\infty \leq 1$.

Theorem 9.1. *Fix an integer $L \in \mathbb{N}^*$ and assume that the following bound holds for any pair of disjoint balls $B_L(x), B_L(y)$ and some quantity $\varsigma(L) > 0$:*

$$\mathbb{P} \{ \exists E \in I : B_L(x) \text{ and } B_L(y) \text{ are } (E, m)\text{-S} \} \leq \varsigma(L).$$

Then for any $x, y \in \mathcal{Z}$ with $d(x, y) > 2L + 1$, any finite connected subgraph (of \mathcal{Z}) $\mathcal{G} \supset B_L(x) \cup B_L(y)$ and any Borel function $\phi \in \mathcal{B}_1(I)$

$$\mathbb{E} [|\langle \mathbf{1}_x | \phi(H_{\mathcal{G}}(\omega)) | \mathbf{1}_y \rangle|] \leq 4e^{-mL} + \varsigma(L). \quad (9.1)$$

Proof. Fix points $x, y \in \mathcal{Z}$ with $d(x, y) > 2L + 1$ and a graph $\mathcal{G} \supset B_L(x) \cup B_L(y)$. The operator $H_{\mathcal{G}}(\omega)$ has a finite orthonormal eigenbasis $\{\psi_i\}$ with respective eigenvalues $\{\lambda_i\}$. Let $\mathbf{S} = \partial B_L(x) \cup \partial B_L(y)$ (recall: this is a set of *pairs* (u, u')); note that $|\mathbf{S}| \leq 2C_D^2 L^D$, by (2.1). Suppose that for some ω , for each i there is $z_i \in \{x, y\}$ such that $B_L(z_i)$ is (λ_i, m) -NS; let $\{v_i\} = \{x, y\} \setminus \{z_i\}$. Denote $\mu_{x,y}(\phi) = |\langle \mathbf{1}_x | \phi(H_{\mathcal{G}}(\omega)) | \mathbf{1}_y \rangle|$, with $\mu_{x,y}(\phi) \leq 1$. Then by the GRI for the eigenfunctions,

$$\begin{aligned} \mu_{x,y}(\phi) &\leq \|\phi\|_\infty \sum_{\lambda_i \in I} |\psi_i(x)\psi_i(y)| \leq \sum_{\lambda_i \in I} |\psi_i(z_i)\psi_i(v_i)| \\ &\leq \sum_{\lambda_i \in I} |\psi_i(v_i)| e^{-mL} (C_D^2 L^D)^{-1} \sum_{(u, u') \in \partial B_L(z_i)} |\psi_i(u)| \\ &\leq e^{-mL} \sum_{\lambda_i \in I} (C_D^2 L^D)^{-1} \sum_{(u, u') \in \mathbf{S}} |\psi_i(u)| (|\psi_i(x)| + |\psi_i(y)|) \\ &\leq e^{-mL} \frac{|\mathbf{S}|}{C_D^2 L^D} \max_{u \in \mathcal{G}} \sum_{\lambda_i \in I} \frac{1}{2} (|\psi_i(u)|^2 + |\psi_i(x)|^2 + |\psi_i(y)|^2) \end{aligned}$$

(using Bessel's inequality and $|\mathbf{S}| \leq 2C_D^2 L^D$)

$$\leq e^{-mL} 2 \max_{u \in \mathcal{G}} (2\|\mathbf{1}_u\|^2 + \|\mathbf{1}_x\|^2 + \|\mathbf{1}_y\|^2) = 4e^{-mL}.$$

Denote $\mathcal{S}_L = \{ \exists E \in I : B_L(x) \text{ and } B_L(y) \text{ are } (E, m)\text{-S} \}$, with $\mathbb{P} \{ \mathcal{S}_L \} \leq \varsigma(L)$, by assumption. Now we conclude:

$$\mathbb{E} [\mu_{x,y}(\phi)] = \mathbb{E} [\mathbf{1}_{\mathcal{S}_L} \mu_{x,y}(\phi)] + \mathbb{E} [\mathbf{1}_{\mathcal{S}_L^c} \mu_{x,y}(\phi)] \leq \varsigma(L) + 4e^{-mL}. \quad \square$$

9.2. Dynamical localization on the entire graph. For the reader's convenience, we repeat below a simple argument, described in [15] and employed earlier by Aizenman et al. [1, 2]. The quantities $\mu_{x,y}^{(H)}(\phi) = \langle \mathbf{1}_x | \phi(H) | \mathbf{1}_y \rangle$ defined, for example, for bounded continuous or Borel functions ϕ , generate signed (i.e., not necessarily positive) spectral measures associated with a self-adjoint operator H :

$$\int d\mu_{x,y}^{(H)}(E) \phi(E) := \langle \mathbf{1}_x | \phi(H) | \mathbf{1}_y \rangle.$$

In particular, we can consider, with $x, y, u \in \mathcal{Z}$ fixed, measures $\mu_{x,y}^k$ related to operators $H_{B_{L_k}(u)}$, for all $k \geq 0$, as well as their counterparts $\mu_{x,y}$ for the operator H on the entire graph \mathcal{Z} . A sufficient condition for the vague convergence $\mu_{x,y}^k \rightarrow \mu_{x,y}$ as $k \rightarrow \infty$ is the strong resolvent convergence $H_{B_{L_k}(u)} \rightarrow H$. Such convergence is well-known to occur for a very large class of operators, including (unbounded) Schrödinger operators in Euclidean spaces and their analogs on the so-called quantum graphs. Indeed, for (not necessarily bounded) operators H_n with a common core \mathcal{D} to converge to an operator H with the same core, it suffices that $H_k \psi \rightarrow H \psi$ strongly for any element $\psi \in \mathcal{D}$ (cf. [32]). For finite-volume operators, one can usually find an appropriate core \mathcal{D} formed by compactly supported functions ψ ; for finite-difference Hamiltonians on graphs (even unbounded, e.g., for DSO with unbounded potentials) one can choose as \mathcal{D} the subset of all functions with finite supports. On such functions, $H_{B_{L_k}(u)} \psi \rightarrow H \psi$ as $k \rightarrow \infty$ (by stabilization), therefore, the spectral measures converge vaguely: $\mu_{x,y}^k \rightarrow \mu_{x,y}$. By Fatou lemma, for any bounded Borel set $A \subset \mathbb{R}$, one has

$$|\mu_{x,y}(A)| \leq \liminf_{k \rightarrow \infty} |\mu_{x,y}^k(A)|$$

(here $|\mu(A)| := \sup\{|\mu(\phi)|, \|\phi\| \leq 1, \text{supp } \phi \subset A\}$). Taking the expectation and using the uniform upper bounds on EF correlators in finite balls, we conclude that

$$\mathbb{E} \left[\sup_{\phi \in \mathcal{B}_1} |\langle \mathbf{1}_x | \phi(H(\omega)) | \mathbf{1}_y \rangle| \right] \leq C e^{-a \ln^{1+c} d(x,y)} \quad (9.2)$$

(using the inequality $L_k^{-\kappa(1+\theta)^k} \leq C e^{-a \ln^{1+c} L_k}$, for some $C, a, c > 0$). In particular, with functions $\phi_t : \lambda \mapsto e^{-it\lambda}$, we obtain the strong dynamical localization property for the ensemble of random Hamiltonians $H(\omega)$.

Taking into account Theorem 8.6, we come to the following sufficient conditions of strong N -particle dynamical localization in an interval $I \subseteq \mathbb{R}$:

Theorem 9.2. *Assume (W1) and suppose that, for some $N \geq 2$ and all $k \geq 0$, $S(I, N, k)$ holds true. Then there are constants $a, c \in (0, +\infty)$ such that for all $\mathbf{x} \in \mathcal{Z}^N$, some $C(\mathbf{x}) \in (0, +\infty)$ and all $\mathbf{y} \in \mathcal{Z}^N$*

$$\mathbb{E} \left[\sup_{t \in \mathbb{R}} |\langle \mathbf{1}_{\mathbf{x}} | e^{-it\mathbf{H}^{(N)}(\omega)} P_I(\mathbf{H}^{(N)}(\omega)) | \mathbf{1}_{\mathbf{y}} \rangle| \right] \leq C(\mathbf{x}) e^{-a \ln^{1+c} \rho(\mathbf{x}, \mathbf{y})}. \quad (9.3)$$

Here and below, $P_I(\mathbf{H}^{(N)}(\omega))$ stands for the spectral projection of operator $\mathbf{H}^{(N)}(\omega)$ on the interval I . Recall that, formally, we assumed in Sections 7–9 the interval I to be finite. In the case of a bounded random potential, the spectrum of the operator is covered by a finite, non-random interval I , so that $P_I(\mathbf{H}^{(N)}(\omega)) = \mathbf{H}^{(N)}(\omega)$.

Theorem 9.3. *Assume (W2) and suppose that, for some $N \geq 2$ and all $k \geq 0$, $S(I, N, k)$ holds true. Then there are constants $C, a, c \in (0, +\infty)$ such that for all $\mathbf{x}, \mathbf{y} \in \mathcal{Z}^N$*

$$\mathbb{E} \left[\sup_{t \in \mathbb{R}} |\langle \mathbf{1}_{\mathbf{x}} | e^{-it\mathbf{H}^{(N)}(\omega)} P_I(\mathbf{H}^{(N)}(\omega)) | \mathbf{1}_{\mathbf{y}} \rangle| \right] \leq C e^{-a \ln^{1+c} \rho_S(\mathbf{x}, \mathbf{y})}. \quad (9.4)$$

10. APPENDIX

10.1. Proof of Lemma 6.4. The ball $\mathbf{B} := \mathbf{B}_{L_k}(\mathbf{u})$ is assumed PI, so there is a decomposition of the configuration \mathbf{u} into two non-interacting subconfigurations, $\mathbf{u} = (\mathbf{u}', \mathbf{u}'')$, so that $\mathbf{U}(\mathbf{u}) = \mathbf{U}(\mathbf{u}') + \mathbf{U}(\mathbf{u}'')$, and the operator $\mathbf{H}_{\mathbf{B}}^{(N)}$ reads as follows:

$$\mathbf{H}_{\mathbf{B}_{L_k}^{(N)}(\mathbf{u})} = \mathbf{H}_{\mathbf{B}_{L_k}^{(n')}(\mathbf{u}')} \otimes \mathbf{1}^{(n'')} + \mathbf{1}^{(n')} \otimes \mathbf{H}_{\mathbf{B}_{L_k}^{(n'')}(\mathbf{u}'')} \quad (10.1)$$

thus its eigenvalues are the sums $E_{a,b} = \lambda_a + \mu_b$, where $\{\lambda_a\} = \Sigma(\mathbf{H}_{\mathbf{B}_{L_k}(\mathbf{u}')}^{(R)})$ is the spectrum of $\mathbf{H}_{\mathbf{B}_{L_k}(\mathbf{u}')}^{(R)}$ and, respectively, $\{\mu_b\} = \Sigma(\mathbf{H}_{\mathbf{B}_{L_k}(\mathbf{u}'')}^{(R)})$. Eigenvectors of $\mathbf{H}_{\mathbf{u},k}^{(R)}$ can be chosen in the form $\Psi_{a,b} = \phi_a \otimes \psi_b$ where $\{\phi_a\}$ are eigenvectors of $\mathbf{H}_{\mathbf{B}_{L_k}^{(n')}(\mathbf{u}')}^{(R)}$ and $\{\psi_b\}$ are eigenvectors of $\mathbf{H}_{\mathbf{B}_{L_k}^{(n'')}(\mathbf{u}'')}^{(R)}$. For each pair (λ_a, μ_b) , the non-resonance assumption $|E - (\lambda_a + \mu_b)| \geq e^{L_k^\beta}$ reads as $|(E - \lambda_a) - \mu_b| \geq e^{L_k^\beta}$ and also as $|(E - \mu_b) - \lambda_a| \geq e^{L_k^\beta}$. Set $\mathbf{G}(E) = (\mathbf{H}_{\mathbf{B}_{L_k}^{(N)}(\mathbf{u})} - E)^{-1}$; then we can write

$$\mathbf{G}(\mathbf{u}, \mathbf{y}; E) = \sum_{\lambda_a} \sum_{\mu_b} \frac{\phi_a(\mathbf{u}') \phi_a(\mathbf{y}') \psi_b(\mathbf{u}'') \psi_b(\mathbf{y}'')}{(\lambda_a + \mu_b) - E} \quad (10.2)$$

$$= \sum_{\lambda_a} \mathbf{P}'_a(\mathbf{u}', \mathbf{y}') \mathbf{G}_{\mathbf{B}_{L_k}(\mathbf{u}'')}(\mathbf{u}'', \mathbf{y}''; E - \lambda_a) \quad (10.3)$$

$$= \sum_{\mu_b} \mathbf{P}''_b(\mathbf{u}'', \mathbf{y}'') \mathbf{G}_{\mathbf{B}_{L_k}(\mathbf{u}')}(\mathbf{u}', \mathbf{y}'; E - \mu_b), \quad (10.4)$$

By assumption, for all $\mu_b \in \sigma(\mathbf{H}_{\mathbf{B}_{L_k}(\mathbf{u}'')}^{(R)})$, the projection ball $\mathbf{B}_{L_k}(\mathbf{u}'')$ is (μ_b, m) -NS, and for all $\lambda_a \in \sigma(\mathbf{H}_{\mathbf{B}_{L_k}(\mathbf{u}')}^{(R)})$, the projection ball $\mathbf{B}_{L_k}(\mathbf{u}')$ is (λ_a, m) -NS.

For any $\mathbf{y} \in \partial^- \mathbf{B}_{L_k}(\mathbf{u})$, either $\rho(\mathbf{u}', \mathbf{y}') = L_k$, in which case we infer from (10.4), combined with (μ_b, m) -NS property of the ball $\mathbf{B}_{L_k}(\mathbf{u}')$, that

$$|\mathbf{G}(\mathbf{u}, \mathbf{y}; E)| \leq |\mathbf{B}_{L_k}(\mathbf{u}'')| e^{-\gamma(m, L_k, N-1)L_k + 2L_k^\beta} \quad (10.5)$$

or $\rho(\mathbf{u}'', \mathbf{y}'') = L_k$, and then we have by (10.3)

$$|\mathbf{G}(\mathbf{u}, \mathbf{y}; E)| \leq |\mathbf{B}_{L_k}(\mathbf{u}')| e^{-\gamma(m, L_k, N-1)L_k + 2L_k^\beta}.$$

In either case, the LHS is bounded by

$$\exp \left(-m(1 + L_k^{-\tau})^{\hat{N} - (N-1)+1} L_k + 2L_k^\beta + \text{Const} \ln L_k \right) < \frac{1}{2} e^{-\gamma(m, L_k, N)},$$

for L_0 large enough, since $m \geq 1$ and

$$L_k^{1-\tau} \equiv L_k^{7/8} \gg L_k^{1/2} + \text{Const} \ln L_k \equiv L_k^\beta + \text{Const} \ln L_k.$$

□

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